

# MONOIDAL CATEGORIES OF COMODULES FOR COQUASI HOPF ALGEBRAS AND RADFORD'S FORMULA

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**ABSTRACT.** We study the basic monoidal properties of the category of Hopf modules for a coquasi Hopf algebra. In particular we discuss the so called fundamental theorem that establishes a monoidal equivalence between the category of comodules and the category of Hopf modules. We present a categorical proof of Radford's  $S^4$  formula for the case of a finite dimensional coquasi Hopf algebra, by establishing a monoidal isomorphism between certain double dual functors.

*Dedicated to I. Shestakov on the occasion of his 60th birthday*

## 1. INTRODUCTION

The main purpose of this paper is to study the monoidal category of Hopf modules for a coquasi Hopf algebra. As a consequence we obtain a proof of Radford's  $S^4$  formula valid for finite dimensional coquasi Hopf algebras. Inspired in [8] we show that this formula is intimately related to the existence of certain natural transformation relating the left and the right double dual functors for the category of right  $H$ -comodules. This natural transformation comes from the application of the structure theorem for Hopf modules, to  ${}^*H$  viewed as a right  $H$ -Hopf module.

Coquasi Hopf algebras are the dual notion of the quasi Hopf algebras defined in [7]. The main difference with Hopf algebras is that for coquasi Hopf algebras the role of the multiplicative and comultiplicative structures is not longer interchangeable. In a coquasi Hopf algebra the multiplicative structure is no longer one dimensional, but two dimensional; this is expressed in the fact that the multiplication is not longer associative but only up to isomorphism, provided by a functional  $\phi$ . The antipode is also defined as a two dimensional structure, the extra dimension provided by two functionals  $\alpha, \beta$ . See below.

The category of Hopf modules in the context of (co)quasi Hopf algebras has been considered by different authors and it was initially studied in [10, 22].

In the case of Hopf algebras, Radford's formula for  $S^4$  was first proved in full generality in [20], with predecessors in [17] and [24]. A more recent proof, appears in [23]. There are many generalizations of the formula from the case of Hopf algebras to other situations, *e.g.*: braided Hopf algebras, bF algebras – braided and classical –, quasi Hopf algebras, weak Hopf algebras, and Hopf algebras over rings. The following is a partial list of references for some proofs of these generalizations: [2], [6], [9], [10], [13], [14] and [19]. Closer to the spirit of our paper an analogue of Radford's formula for finite tensor categories appears in [8].

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In this Introduction, after a general description of the paper, we briefly recall the definition and the first basic properties of a coquasi bialgebra and of a coquasi Hopf algebra. We define the notion of monoidal morphism that is the adequate notion of morphism between coquasi bialgebras. Later in this Introduction we establish the basic notations that will be used along the paper.

In Section 2, that is the technical core of the paper, we present the basic properties of the monoidal categories used later. We work with the categories of comodules and Hopf modules for coquasi bialgebras and recall the properties of the monoidal structures induced by the cotensor product over  $H$  and by the tensor product over  $\mathbb{k}$ . We look at some basic monoidal functors associated to monoidal morphisms given by corestriction of scalars and its adjoints given by coinduction. We also consider other monoidal functors –*e.g.*, the left adjoint comodule and right adjoint comodule functors– that will be used later.

In Section 3 we recall that in the case of coquasi Hopf algebras with invertible antipode the tensor categories of finite dimensional comodules –or bicomodules– are rigid, *i.e.*, each object has a left and a right dual. We use this rigidity in order to describe explicitly –for finite dimensional Hopf algebras– the monoidal structure of the antipode.

In Section 4 we present a proof of the version of the fundamental theorem on Hopf modules for coquasi Hopf algebras that we need later in the paper. By applying some general results on Hopf modules over autonomous pseudomonoids to our context we prove that the free right Hopf module functor is a monoidal equivalence from the category of comodules into the category of Hopf modules.

In Section 5, we apply the fundamental theorem on Hopf modules to  ${}^*H$  and obtain the Frobenius isomorphism that is a morphism in the category of Hopf modules between  $H$  and  ${}^*H$ . Along the way we identify the one dimensional object of cointegrals in this context.

In Section 6, using the categorical machinery constructed above and in the same vein than in [8], we prove the existence of a natural monoidal isomorphism between the double duals on the left and on the right of a finite dimensional left  $H$ –module. This isomorphism will yield Radford’s formula.

It is not obvious *a priori* that the formula obtained for finite dimensional coquasi Hopf algebras is related to the classical Radford’s formula for Hopf algebras. Thus, in Section 7, we apply the previously developed techniques to the case that  $H$  is a classical Hopf algebra in order to deduce the original Radford’s formula for  $S^4$  –see [20]–.

In Section 8 we present in an Appendix the categorical background needed to prove some of the basic monoidal properties of the cotensor product. We recall a few basic definitions and results about density of functors and completions of categories under certain classes of colimits.

**1.1. Basic definitions.** Next we summarize the basic definitions that we need.

Recall that the category of coalgebras and morphisms of coalgebras has a monoidal structure such that the forgetful functor into the category of vector spaces is monoidal. In other words, the tensor product over  $\mathbb{k}$  of two coalgebras is a coalgebra and  $\mathbb{k}$  is a coalgebra, in a canonical way. This is a consequence of the fact of that category of vector spaces is braided, and in fact symmetric, with the usual switch  $\text{sw} : V \otimes W \rightarrow W \otimes V$ .

Assume  $(C, \Delta, \varepsilon)$  is a coalgebra. The maps  $\Delta : C \rightarrow C \otimes C$  and  $\varepsilon : C \rightarrow \mathbb{k}$  are the comultiplication and the counit respectively. We will use Sweedler's notation as introduced in [25], and write  $\Delta(c) = \sum c_1 \otimes c_2$ . We use the notation  $\Delta^2$  for the morphism  $\Delta^2(c) = \sum c_1 \otimes c_2 \otimes c_3$ . Moreover, the convolution product will be denoted by the symbol  $\star$ .

**Definition 1.** A *coquasi bialgebra* structure on the coalgebra  $(C, \Delta, \varepsilon)$  is a triple  $(p, u, \phi)$  where  $p : C \otimes C \rightarrow C$  –the product– and  $u : \mathbb{k} \rightarrow C$  –the unit– are coalgebra morphisms, and  $\phi : C \otimes C \otimes C \rightarrow \mathbb{k}$  –the associator– is a convolution-invertible functional, satisfying the following axioms.

$$p(u \otimes \text{id}) = \text{id} = p(\text{id} \otimes u) \quad (1)$$

$$\sum (c_1 d_1) e_1 \phi(c_2 \otimes d_2 \otimes e_2) = \sum \phi(c_1 \otimes d_1 \otimes e_1) c_2 (d_2 e_2) \quad (2)$$

$$\begin{aligned} \sum \phi(c_1 d_1 \otimes e_1 \otimes f_1) \phi(c_2 \otimes d_2 \otimes e_2 f_2) = \\ = \sum \phi(c_1 \otimes d_1 \otimes e_1) \phi(c_2 \otimes d_2 e_2 \otimes f_1) \phi(d_3 \otimes e_3 \otimes f_2) \end{aligned} \quad (3)$$

$$\phi(c \otimes 1 \otimes d) = \varepsilon(c) \varepsilon(d) \quad (4)$$

The quadruple  $(C, p, u, \phi)$  is called a *coquasi bialgebra*.

Along this paper coalgebras and coquasi bialgebras will be denoted with the letters,  $C, D$ , etc.

The dual concept of a quasi bialgebra was originally defined in [7].

In the above equations we have written  $1 \in C$  for the image under  $u$  of the unit of  $\mathbb{k}$ , and  $p(c, d) = cd$ . Moreover, when multiplying three elements of  $C$  we used parenthesis in order to establish the way we performed the operations.

The equation (2) can be interpreted in a precise way as a naturality condition on  $\phi$ . Equations (3) and (4) can be written as the equalities  $\phi(p \otimes \text{id} \otimes \text{id}) \star \phi(\text{id} \otimes \text{id} \otimes p) = (\phi \otimes \varepsilon) \star \phi(\text{id} \otimes p \otimes \text{id}) \star (\varepsilon \otimes \phi)$  and  $\phi(\text{id} \otimes u \otimes \text{id}) = \varepsilon \otimes \varepsilon$  valid in the convolution algebras  $(C \otimes C \otimes C)^\vee$  and  $(C \otimes C)^\vee$  respectively.

Applying equation (3) to the case of  $c = d = 1$ , we obtain that  $\phi(1 \otimes e \otimes f) = \phi(1 \otimes e_1 \otimes f_1) \phi(1 \otimes e_2 \otimes f_2)$ . Hence  $\rho : C \otimes C \rightarrow \mathbb{k}$ ,  $\rho(e \otimes f) = \phi(1 \otimes e \otimes f)$  is convolution invertible and  $\rho \star \rho = \rho$ . Then  $\rho = \varepsilon \otimes \varepsilon$  and for later use we record below this and other similar consequence of the axioms of a coquasi Hopf algebra.

$$\phi(1 \otimes c \otimes d) = \varepsilon(c) \varepsilon(d) \quad \phi(c \otimes d \otimes 1) = \varepsilon(c) \varepsilon(d) \quad (5)$$

**Observation 1.** If  $(C, p, u, \phi)$  is a coquasi bialgebra, then  $C^{\text{cop}}$  has a structure of a coquasi bialgebra with unit  $u$ , multiplication  $p \text{ sw}$  and associator  $\phi(\text{id} \otimes \text{sw})(\text{sw} \otimes \text{id})(\text{id} \otimes \text{sw})$ . We shall denote this coquasi bialgebra by  $C^\circ$ . In the literature  $C^\circ$  is denoted by  $C^{\text{copop}}$ .

Next we define the concept of *monoidal morphism* between coquasi bialgebras. Monoidal morphisms are to coquasi bialgebras what bialgebra morphisms are to bialgebras.

The monoidal morphisms are the adequate kind of morphisms for our category as they preserve multiplication and unit up to coherent isomorphisms. Although we will only need this concept of morphism, for the sake of clarification we also give the definition of *lax monoidal* morphism as in this general case the role of the invertible scalar  $\rho$  appearing in the definition below is more transparent.

**Definition 2.** Let  $C$  and  $D$  be coquasi bialgebras and  $f : C \rightarrow D$  be a morphism of coalgebras. A *lax monoidal structure* on  $f$  is a functional  $\chi : C \otimes C \rightarrow \mathbb{k}$  and a scalar  $\rho \in \mathbb{k}$  satisfying

$$(\chi \otimes p(f \otimes f))\Delta_{C \otimes C} = (fp \otimes \chi)\Delta_{C \otimes C} \quad u = fu \quad (6)$$

$$(\phi_D(f \otimes f \otimes f)) \star (\chi \otimes \varepsilon) \star (\chi(p \otimes \text{id})) = (\varepsilon \otimes \chi) \star (\chi(\text{id} \otimes p)) \star \phi_C \quad (7)$$

$$\rho\chi(u \otimes \text{id}) = \varepsilon = \rho\chi(\text{id} \otimes u). \quad (8)$$

The lax monoidal structure  $(\chi, \rho)$  is called a *monoidal structure* when  $\chi$  is invertible, – notice that  $\rho$  is always invertible–. A morphism of coalgebras between coquasi bialgebras equipped with a (lax) monoidal structure is called a *(lax) monoidal morphism*.

A monoidal morphism between two coquasi bialgebras is a morphism of coalgebras equipped with a monoidal structure.

The above terminology on monoidal structures comes from category theory. In fact, the concept of monoidal morphism as defined above is a special instance of the concept of monoidal 1-cell between pseudomonoids.

In particular, in the case that the monoidal structure is  $\chi = \varepsilon \otimes \varepsilon$  and  $\rho = 1$  equations (6), (7) and (8) simply say that  $f$  preserves the product, the unit and that  $\phi_D(f \otimes f \otimes f) = \phi_C$ . Hence, it is clear in particular that a map of coquasi bialgebras that preserves the product, the coproduct and the associator, is a monoidal morphism.

The definition above can be generalized to the concept of a monoidal  $(C, D)$ –bicomodule, where  $C$  and  $D$  are coquasi bialgebras. In that case,  $f : C \rightarrow D$  is a monoidal morphism if and only if the bicomodule  $f_+$  (see Definition 4) is a monoidal bicomodule (see Theorem 2). In this paper we will not cover these general aspects of the theory.

**Observation 2.** If  $f : C \rightarrow D$  and  $g : D \rightarrow E$  are monoidal morphisms with monoidal structures  $(\chi^f, \rho^f)$  and  $(\chi^g, \rho^g)$  respectively, then  $gf$  has canonical monoidal structure, namely,  $(\chi^g(f \otimes f) \star \chi^f, \rho^f \rho^g)$ . Also, the identity morphism  $\text{id} : C \rightarrow C$  is equipped with a monoidal structure given by  $(\varepsilon \otimes \varepsilon, 1)$ .

In Proposition 3 we show that the antipode  $S$  –see the definition below– of a finite dimensional coquasi Hopf algebra  $H$  is a monoidal morphism from  $H^{\text{copop}}$  to  $H$ .

**Definition 3.** An *antipode* for the coquasi bialgebra  $H$  is a triple  $(S, \alpha, \beta)$  where  $S : H^{\text{cop}} \rightarrow H$  is a coalgebra morphism and the functionals  $\alpha, \beta : H \rightarrow \mathbb{k}$  satisfy the following equations.

$$\sum S(h_1)\alpha(h_2)h_3 = \alpha(h)1 \quad \sum h_1\beta(h_2)S(h_3) = \beta(h)1 \quad (9)$$

$$\sum \phi^{-1}(h_1 \otimes Sh_3 \otimes h_5)\beta(h_2)\alpha(h_4) = \varepsilon(h) \quad (10)$$

$$\sum \phi(Sh_1 \otimes h_3 \otimes Sh_5)\alpha(h_2)\beta(h_4) = \varepsilon(h) \quad (11)$$

A *coquasi Hopf algebra* is a coquasi bialgebra equipped with an antipode.

Along this paper coquasi Hopf algebras will be denoted as  $H$ .

**Observation 3.** If  $(S, \alpha, \beta)$  is an antipode for the coquasi bialgebra  $H$  then  $(S, \beta, \alpha)$  is an antipode for the coquasi bialgebra  $H^\circ$  considered in Observation 1.

**Observation 4.** For future use we record the following fact. If  $a \in H$  is a group like element then  $S(a) = a^{-1}$ . Indeed, from the equality  $\phi(Sa \otimes a \otimes Sa)\alpha(a)\beta(a) = 1$ , we deduce that  $\alpha(a) \neq 0$ . Then, from the equality  $S(a)\alpha(a)a = \alpha(a)1$  we deduce that  $S(a)a = 1$ . The equality  $aS(a) = 1$  can be proved in a similar manner.

Moreover, if  $b \in H$  satisfies that  $ab = ba = 1$ , then  $Sa = (ba)Sa$ . Reassociating the product, if we call  $\gamma : H \rightarrow \mathbb{k}$  the functional  $\gamma(x) = \phi(x \otimes a \otimes Sa)$ , we have that:  $Sa = (ba)Sa = \gamma^{-1} \rightharpoonup b \leftarrow \gamma$ . Then  $b = \gamma \rightharpoonup Sa \leftarrow \gamma^{-1}$  and since  $Sa$  is a group like element,  $b = \gamma \rightharpoonup Sa \leftarrow \gamma^{-1} = Sa$ .

In the particular case of the group like element 1, we have that  $S(1) = 1$ .

The definition of coquasi bialgebra (or rather its dual concept of quasi bialgebra) was introduced by Drinfel'd in [7]. The crucial observation was that in order to guarantee the corresponding module category to be monoidal, the associativity of the coproduct was only necessary up to conjugation. The concept of antipode (called by many authors a quasi antipode) and of quasi Hopf algebra as defined in [7], is needed in order guarantee the existence of duals in the corresponding categories of finite dimensional objects.

Along this paper we study the basic properties of the categories of modules and comodules for a coquasi Hopf algebra. These categories have been considered by many authors –see for example [15] and more specifically [10] and [22].

Our main interest lays in the case that the coquasi Hopf algebra is finite dimensional as a vector space. In this case, it is known (see [4] and [22]) that the antipode  $S$  is a bijective linear transformation. The composition inverse of  $S$  will be denoted as  $\overline{S}$ .

We finish this Introduction by describing some of the notations we use.

We denote the usual duality functor in the category of vector spaces as  $V \mapsto V^\vee$  and the usual evaluation and coevaluation maps as  $e$  and  $c$ .

Let  $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{k}, \Phi, l, r)$  be a monoidal category with monoidal structure  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , unit object object  $\mathbb{k}$ , associativity constraint with components  $\Phi_{M,N,L} : (M \otimes N) \otimes L \rightarrow M \otimes (N \otimes L)$  and left and right unit constraints  $l$  and  $r$ . We denote as  $\mathcal{C}^{\text{rev}}$  the tensor category  $(\mathcal{C}, \otimes^{\text{rev}} = \otimes^{\text{sw}}, \mathbb{k}, \hat{\Phi}, \hat{r}, \hat{l})$  where  $\otimes^{\text{rev}}(M \otimes N) = N \otimes M$ ,  $\hat{\Phi}_{M,N,L} = \Phi_{L,N,M}^{-1}$ ,  $\hat{r} = l$  and  $\hat{l} = r$ .

Let  $\mathcal{C}$  and  $\mathcal{D}$  be monoidal categories and  $T : \mathcal{C} \rightarrow \mathcal{D}$  a functor. A monoidal structure on  $T$  is a natural isomorphism  $\otimes(T \times T) \Rightarrow T \otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$  and an isomorphism  $\mathbb{k} \rightarrow T(\mathbb{k})$  satisfying a certain natural list of coherence axioms (see [12] for details). A monoidal functor is a functor equipped with a monoidal structure.

We assume that the reader is familiar with the basic concepts concerning rigidity for tensor categories as presented for example in [12] or [15]. Recall that monoidal functors preserve duals. In other words, if  $T : \mathcal{C} \rightarrow \mathcal{D}$  is a monoidal functor and  $M \in \mathcal{C}$  is a left rigid object in  $\mathcal{C}$ , then  $T(M)$  is also left rigid and there is a canonical natural isomorphism  $\eta : T(*M) \rightarrow *T(M)$ . This isomorphism is the unique arrow such that makes the diagram below commutative

$$\begin{array}{ccc}
 T(*M) \otimes T(M) & \xrightarrow{a} & T(*M \otimes M) \xrightarrow{T(\text{ev}_M)} T(\mathbb{k}) \\
 \eta \otimes \text{id} \downarrow & & \downarrow b \\
 *T(M) \otimes T(M) & \xrightarrow{\text{ev}_{T(M)}} & \mathbb{k}
 \end{array}$$

where the maps  $a$  and  $b$  are the maps given by the monoidal structure of  $T$ .

Also, following the standard usage we write  ${}^C\mathcal{M}$ ,  $\mathcal{M}^D$  and  ${}^C\mathcal{M}^D$  for the categories of left  $C$ -comodules, right  $D$ -comodules and  $(C, D)$ -bicomodules, where  $C$  and  $D$  are coalgebras. For the coactions associated to the objects in these categories we also use Sweedler's notation. In these situations, when we add the subscript  $f$  to the symbols, we mean to say that we are restricting our attention to the subcategories of finite-dimensional objects, *e.g.*,  ${}^C\mathcal{M}_f^D$  is the full subcategory of  ${}^C\mathcal{M}^D$  whose objects are finite dimensional  $\mathbb{k}$ -spaces.

## 2. THE CATEGORIES OF BICOMODULES AND OF HOPF MODULES

**2.1. The cotensor product.** We start by briefly reviewing some of the basic properties of the cotensor product. Given coalgebras  $C, D$  and  $E$ , one can define the *cotensor product* functor  ${}^C\mathcal{M}^D \times {}^D\mathcal{M}^E \rightarrow {}^C\mathcal{M}^D$  as follows. If  $M$  and  $N$  are objects of  ${}^C\mathcal{M}^D$  and  ${}^D\mathcal{M}^E$  respectively, its cotensor product over  $D$ , denoted by  $M \square_D N$  is the equalizer of the following diagram

$$M \otimes N \xrightleftharpoons[\text{id} \otimes ((\text{id} \otimes \text{id} \otimes \varepsilon)\chi_N)]{((\varepsilon \otimes \text{id} \otimes \text{id})\chi_M) \otimes \text{id}} M \otimes D \otimes N$$

endowed with the bicomodule structure induced by the left coaction of  $M$  and the right coaction of  $N$ .

If  $F$  is another coalgebra, there is a natural isomorphism between the two obvious functors  ${}^C\mathcal{M}^D \times {}^D\mathcal{M}^E \times {}^E\mathcal{M}^F \rightarrow {}^C\mathcal{M}^F$ , with components  $L \square_D (M \square_E N) \cong (L \square_D M) \square_E N$  induced by the universal property of the equalizers. Also, the functors  $C \square_C -$  and  $-\square_D D : {}^C\mathcal{M}^D \rightarrow {}^C\mathcal{M}^D$  are canonically isomorphic to the identity functor. All these data satisfy coherence conditions; in categorical terminology we say that the categories of bicomodules form a *bicategory* [1].

From the above, it is clear that the cotensor product provides a monoidal structure to  ${}^C\mathcal{M}^C$ , with unit object  $(C, \Delta^2)$ .

**Observation 5.** The cotensor product functor  ${}^C\mathcal{M}^D \times {}^D\mathcal{M}^E \rightarrow {}^C\mathcal{M}^D$  preserves filtered colimits in each variable. This is because finite limits commute with finite colimits.

Next we consider corestriction functors.

**Definition 4.** If  $f : C \rightarrow D$  is a morphism of coalgebras, we shall denote by  $f_+ = C_f \in {}^C\mathcal{M}^D$  the object obtained from the regular bicomodule  $C$  by corestriction with  $f$  on the right, *i.e.*, the coaction in  $f_+ = C_f$  is given by  $x \mapsto \sum x_1 \otimes x_2 \otimes f(x_3)$ . Similarly, we shall denote by  $f^+ = {}_f C \in {}^D\mathcal{M}^C$  the object obtained from  $C$  by corestriction on the left.

Taking cotensor products with bicomodules that are induced by morphisms of coalgebras has convenient properties.

**Observation 6.** Suppose that  $f, C$  and  $D$  are as above and that  $A$  is an arbitrary coalgebra,

- (1) For any bicomodule  $M \in {}^A\mathcal{M}^C$ , with coaction  $\chi_M$ , the cotensor product  $M \square_C f_+ = M \square_C C_f \in {}^A\mathcal{M}^D$  is canonically isomorphic with the bicomodule –sometimes called also  $M_f$ – with underlying space  $M$  and coaction  $(\text{id}_A \otimes \text{id}_M \otimes f)\chi_M : M \rightarrow A \otimes M \otimes D$ .

In a completely analogous way, if  $N \in {}^C\mathcal{M}^A$ , then the cotensor product  $f^+\square_C N = {}_f C \square_C N \in {}^D\mathcal{M}^A$  is canonically isomorphic to the bicomodule –sometimes called  ${}_f N$ – with underlying space  $N$  and coaction  $(f \otimes \text{id}_N \otimes \text{id}_C)\chi_N : N \rightarrow D \otimes N \otimes A$ .

Hence,  $-\square_C f_+ = -\square_C C_f$  and  $f^+\square_C - = {}_f C \square_C -$  are the functors  $M \mapsto M_f : {}^A\mathcal{M}^C \rightarrow {}^A\mathcal{M}^D$  and  $N \mapsto {}_f N : {}^C\mathcal{M}^A \rightarrow {}^D\mathcal{M}^A$  given by corestriction with  $f$ .

- (2) Given a morphism of coalgebras  $f : C \rightarrow D$ , it is clear that the functor considered above  $-\square_C f_+ = -\square_C C_f : {}^A\mathcal{M}^C \rightarrow {}^A\mathcal{M}^D$  is left adjoint to the so called coinduction functor  $-\square_D f^+ = -\square_D {}_f C : {}^A\mathcal{M}^D \rightarrow {}^A\mathcal{M}^C$ .

Similarly, the other corestriction functor  $f^+\square_C - = {}_f C \square_C - : {}^C\mathcal{M}^A \rightarrow {}^D\mathcal{M}^A$  is left adjoint to the so called coinduction functor  $f_+\square_D - = C_f \square_D - : {}^D\mathcal{M}^A \rightarrow {}^C\mathcal{M}^A$ .

- (3) Assume that we have two morphisms of coalgebras  $f : C \rightarrow D$  and  $g : D \rightarrow E$ . In that situation we have canonical isomorphisms between  $(gf)_+ \cong f_+\square_D g_+$  and  $(gf)^+ \cong g^+\square_D f^+$ .

**Definition 5.** Assume that the coalgebra  $D$  has a group like element that we call  $1 \in D$ . We apply the above construction to the morphism of coalgebras  $u : \mathbb{k} \rightarrow D$ . In this case we abbreviate  $(-)_0 = -\square_{\mathbb{k}} u_+ = (-)_u : {}^A\mathcal{M} \rightarrow {}^A\mathcal{M}^D$ . Similarly we call  ${}_0(-) = u^+\square_{\mathbb{k}} - = {}_u(-) : \mathcal{M}^A \rightarrow {}^D\mathcal{M}^A$ .

**Observation 7.** (1) The underlying space for  $M_0$  is  $M$  and the explicit formula for the associated coaction is  $\chi_0(m) = \sum m_{-1} \otimes m_0 \otimes 1$  if  $(M, \chi) \in {}^A\mathcal{M}$ . As we observed before this functor is left adjoint to  $-\square_{\mathbb{k}} u^+ : {}^A\mathcal{M}^D \rightarrow {}^A\mathcal{M}$ . It is clear that if  $M \in {}^A\mathcal{M}^D$ , then  $N\square_{\mathbb{k}} u^+ = N^{\text{co}D}$ . In other words, the functor  $(-)_0$  that produces from a left  $A$ -comodule the  $(A, D)$ -bicomodule with trivial right structure is left adjoint to the fixed point functor.

- (2) The underlying space to  ${}_0 M$  is the same than  $M$  and the coaction is  ${}_0\chi(m) = \sum 1 \otimes m_0 \otimes m_1$ . This functor is left adjoint to  $u_+\square_{\mathbb{k}} - : {}^D\mathcal{M}^A \rightarrow \mathcal{M}^A$ , that is the functor that takes the left coinvariants, i.e., sends  $M \mapsto {}^{\text{co}D}M$ . In other words, the functor  $(-)_0$  that produces from a right  $A$ -comodule the  $(D, A)$ -bicomodule with the trivial left structure is left adjoint to the left fixed point functor.

**Theorem 1.** Let  $g, h : C \rightarrow D$  be two morphisms of coalgebras, and consider the following structures.

- (1) Functionals  $\gamma : C \rightarrow \mathbb{k}$  satisfying  $(\gamma \otimes g)\Delta = (h \otimes \gamma)\Delta$ .
- (2) Morphisms of bicomodules  $\theta : g_+ \rightarrow h_+$ .
- (3) Natural transformations  $\Theta : (-\square_C g_+) \Rightarrow (-\square_C h_+) : \mathcal{M}^C \rightarrow \mathcal{M}^D$ .

Each structure of type (1) induces a structure of type (2) by  $\theta = (\text{id}_C \otimes \gamma)\Delta$  and each structure of type (2) induces a structure of type (3) by  $\Theta = -\square_C \theta$ . Moreover, if  $C$  is finite dimensional these correspondences are bijections, with inverses given by  $\gamma = \varepsilon\theta$  and  $\theta = \Theta_C$ . The identity and the composition of the natural transformations in (3) correspond to the identity and the composition of the morphisms in (2) and to the counit  $\varepsilon$  and the convolution product of the functionals in (1). In particular, the natural transformation  $\Theta$  is invertible iff the associated morphism  $\theta$  is invertible iff the corresponding functional  $\gamma$  is convolution-invertible.

*Proof.* That each structure (1) induces a structure (2) and each structure in (2) induces a structure (3) is easily verified.

Next we prove that the above described maps are indeed a bijection between (1) and (2). The map  $\theta$  satisfies

$$(\text{id} \otimes \theta \otimes g)\Delta^2 = (\text{id} \otimes \text{id} \otimes h)\Delta^2\theta. \quad (12)$$

Composing the above equality with  $\text{id} \otimes \text{id} \otimes \varepsilon$  we deduce that  $\Delta\theta = (\text{id} \otimes \theta)\Delta$ . Now, if we compose Equation (12) with  $\varepsilon \otimes \varepsilon \otimes \text{id}$  we obtain  $(\gamma \otimes g)\Delta = h\theta$ , with  $\gamma = \varepsilon\theta$ . A direct calculation shows that  $(h \otimes \gamma)\Delta = (h \otimes \varepsilon)(\text{id} \otimes \theta)\Delta = (h \otimes \varepsilon)\Delta\theta = h\theta$ . Hence,  $\gamma = \varepsilon\theta$  satisfies condition (1). Clearly, the correspondences given above between elements  $\gamma$  and  $\theta$  are inverses of each other.

If we assume that  $C$  is finite dimensional, the bijection between the structures in (2) and (3) is a consequence of Observation 22.  $\square$

For a functional  $\gamma$  as in Theorem 1.3, we will denote as  $\gamma_+ : g_+ \rightarrow h_+$  the associated morphism of comodules and as  $\Gamma$  the corresponding natural transformation.

**Observation 8.** In the case that  $\gamma$  is convolution invertible, the condition (3) that relates  $g$  and  $h$  in Theorem 1, can be written as  $g = \gamma^{-1} \star h \star \gamma$  or as any of the equalities below valid for all  $c \in C$ :

$$g(c \leftarrow \gamma) = h(\gamma \rightarrow c) \quad , \quad g(c) = h(\gamma \rightarrow c \leftarrow \gamma^{-1}) \quad (13)$$

**2.2. The tensor product over  $\mathbb{k}$ .** When we consider coalgebras that have the additional structure of a coquasi bialgebra, the corresponding categories of comodules and of bicomodules have –besides the monoidal structure given by the cotensor product– another monoidal structure. This monoidal structure is based upon the tensor product over the base field  $\mathbb{k}$  with associativity constraint defined in terms of the corresponding functional  $\phi$ . For example if  $C$  and  $D$  are coquasi bialgebras with associators  $\phi_C$  and  $\phi_D$  respectively, if  $L, M, N \in {}^C\mathcal{M}^D$  the associativity constraint is the map

$$\Phi_{L,M,N} : (L \otimes M) \otimes N \rightarrow L \otimes (M \otimes N)$$

given by the formula

$$\Phi((l \otimes m) \otimes n) = \sum \phi_C(l_{-1} \otimes m_{-1} \otimes n_{-1})l_0 \otimes (m_0 \otimes n_0)\phi_D^{-1}(l_1 \otimes m_1 \otimes n_1) \quad (14)$$

Here we view  $M \otimes N$  as an object in  ${}^C\mathcal{M}^D$  with the usual structure:  $\chi_{M \otimes N}(m \otimes n) = \sum m_{-1}n_{-1} \otimes m_0 \otimes n_0 \otimes m_1n_1 \in C \otimes M \otimes N \otimes D$ . Notice that the above formula for the associativity constraint can be written using the standard actions associated to coactions as follows:

$$\Phi_{L,M,N}((l \otimes m) \otimes n) = \phi_D^{-1} \rightarrow l \otimes m \otimes n \leftarrow \phi_C \in L \otimes (M \otimes N).$$

The unit constraints  $M \otimes \mathbb{k} \cong M$  and  $\mathbb{k} \otimes M \cong M$  are the same than in the category of  $\mathbb{k}$ -vector spaces.

In case that the categories are  ${}^C\mathcal{M}$  or  $\mathcal{M}^C$ , the constraints are defined similarly but using only the action by  $\phi^{-1}$  on the left for  $\mathcal{M}^C$  and of  $\phi$  on the right for  ${}^C\mathcal{M}$ . The monoidal categories  $\mathcal{M}^C$  and  ${}^C\mathcal{M}$  can also be defined as  ${}^{\mathbb{k}}\mathcal{M}^C$  and  ${}^C\mathcal{M}^{\mathbb{k}}$  respectively.



**Observation 9.** If  $C$  is a coquasi bialgebra with unit  $u$  and multiplication  $p$ , the triple  $(C, p, u) \in {}^C\mathcal{M}^C$  is an associative algebra. This can be proved directly using equation (2), which can be rewritten as  $p(\text{id} \otimes p)\Phi_{C,C,C} = p(p \otimes \text{id}) : (C \otimes C) \otimes C \rightarrow C$ .

**Definition 6.** The category of right  $C$ -modules within  ${}^C\mathcal{M}^C$  will be denoted as  ${}^C\mathcal{M}_C^C$  and called the category of Hopf modules. Similarly we define the category  ${}_C\mathcal{M}^C$ .

Notice that the unit object  $\mathbb{k}$  of the monoidal structure  $\otimes$  is canonically a Hopf module with action given by the counit  $\varepsilon$ .

The category of Hopf modules in this context was first considered in [10] and [22].

**Observation 10.** a) The  $\square_C$  monoidal structure of  ${}^C\mathcal{M}^C$  lifts to a monoidal structure on  ${}^C\mathcal{M}_C^C$  in such a way that the forgetful functor  ${}^C\mathcal{M}_C^C \rightarrow {}^C\mathcal{M}^C$  is monoidal.

Indeed, if  $M, N, L, R$  are in  ${}^C\mathcal{M}^C$ , one easily can define –using the universal property of equalizers– a natural morphism of bicomodules  $(M \square_C N) \otimes (L \square_C R) \rightarrow (M \otimes L) \square_C (N \otimes R)$  relating both monoidal structures on  ${}^C\mathcal{M}^C$ . If  $L = R = C$ , we obtain a map  $(M \square_C N) \otimes C \rightarrow (M \otimes C) \square_C (N \otimes C)$  that composed with the right  $C$ -actions on  $M$  and  $N$  – $a_M : M \otimes C \rightarrow M$  and  $a_N : N \otimes C \rightarrow N$ – endows  $M \square_C N$  with the structure of a Hopf module

$$(M \square_C N) \otimes C \rightarrow (M \otimes C) \square_C (N \otimes C) \xrightarrow{a_M \otimes a_N} M \square_C N. \quad (15)$$

Clearly the unit object  $C$  of  $\square_C$  in  ${}^C\mathcal{M}^C$  is also a unit object in  ${}^C\mathcal{M}_C^C$ .

b) If  $f : \mathbb{k} \rightarrow C$  and  $g : C \rightarrow C$  are morphisms of coalgebras –in particular this means that  $f(1) \in C$  is a group like element– then  $f_+ \otimes (M \square_C g_+) \cong M \square_C p(f \otimes g)_+$  and  $(M \square_C g_+) \otimes f_+ \cong M \square_C p(g \otimes f)_+$ .

### 2.3. Monoidal functors induced by monoidal morphisms.

**Theorem 2.** Let  $f : C \rightarrow D$  be a coalgebra morphism, and consider the following structures.

- (1) Monoidal structures on  $f$ ,
- (2) Monoidal structures on the functor  $(-\square_C f_+) : \mathcal{M}^C \rightarrow \mathcal{M}^D$ ,
- (3) Monoidal structures on the functor  $(f^+ \square_C -) : {}^C\mathcal{M} \rightarrow {}^D\mathcal{M}$ .

Each structure (1) induces structures (2) and (3). Moreover, if  $C$  is finite-dimensional there is a bijection between the three types of structures.

*Proof.* First, we consider the relationship of structures of type (1) with structures of type (2). If  $\chi : C \otimes C \rightarrow \mathbb{k}$ ,  $\rho \in \mathbb{k}$  is a monoidal structure on  $f : C \rightarrow D$ , then the transformation with components  $\Theta_{M,N} : M \otimes N \rightarrow M \otimes N$  given by  $\Theta_{M,N}(m \otimes n) = \chi \rightharpoonup (m \otimes n) = \sum \chi(m_1 \otimes n_1)m_0 \otimes n_0$  together with the isomorphism  $\mathbb{k} \rightarrow \mathbb{k}$  given by multiplication by  $\rho$  is a monoidal structure as in (2). Indeed, for example the condition that the map  $\Theta_{M,N} : M_f \otimes N_f \rightarrow (M \otimes N)_f$  is a morphism of  $H$ -comodules –recall the notations of Observation 6– is equivalent with condition (6) in Definition 2. A structure as in (3) is obtained in a similar way, using  $\chi^{-1}$  and  $\rho^{-1}$ .

If we now assume that  $C$  is finite-dimensional we can proceed backwards in order to go from (2) to (1). Let  $\Theta$  be a natural transformation as depicted below.

$$\begin{array}{ccc}
 \mathcal{M}^C \times \mathcal{M}^C & \xrightarrow{(-\square_C f_+) \times (-\square_C f_+)} & \mathcal{M}^D \times \mathcal{M}^D \\
 \otimes \downarrow & \Downarrow \Theta & \downarrow \otimes \\
 \mathcal{M}^C & \xrightarrow{(-\square_C f_+)} & \mathcal{M}^D
 \end{array} \quad (16)$$

Since all the functors in this diagram preserve filtered colimits,  $\Theta$  is determined by its restriction to the categories of finite-dimensional comodules. So we can substitute the categories of comodules in diagram (16) by the corresponding categories of finite-dimensional comodules. In the appendix –Section 8– we prove that for a finite dimensional coalgebra  $C$ , composition with the tensor product functor  $\otimes_{\mathbb{k}} : \mathcal{M}_f^C \times \mathcal{M}_f^C \rightarrow \mathcal{M}_f^{C \otimes C}$  induces an equivalence  $\text{Lex}[\mathcal{M}_f^{C \otimes C}, \mathcal{M}_f^D] \simeq \text{Lex}[\mathcal{M}_f^C, \mathcal{M}_f^C; \mathcal{M}_f^D]$ . The category  $\text{Lex}[\mathcal{M}_f^C, \mathcal{M}_f^C; \mathcal{M}_f^D]$  appearing on the right hand side of the equivalence is the category of functors from  $\mathcal{M}_f^C \otimes \mathcal{M}_f^C$  to  $\mathcal{M}_f^D$  which are left exact in each variable—see also the Appendix for the definition of  $\mathcal{M}_f^C \otimes \mathcal{M}_f^C$ —. Using this fact, we deduce that natural transformations as in diagram (16) are in bijection with natural transformations as in the diagram below:

$$\begin{array}{ccc}
 \mathcal{M}_f^{C \otimes C} & \xrightarrow{-\square_{C \otimes C}(f \otimes f)_+} & \mathcal{M}_f^{D \otimes D} \\
 -\square_{C \otimes C} p_+ \downarrow & \Downarrow \Theta' & \downarrow -\square_{D \otimes D} p_+ \\
 \mathcal{M}_f^C & \xrightarrow{-\square_C f_+} & \mathcal{M}_f^D.
 \end{array}$$

Also these type of natural transformations are in bijective correspondence with bicomodule morphisms  $(p(f \otimes f))_+ \rightarrow (fp)_+$ . These bicomodule morphisms correspond bijectively to functionals  $\chi : C \otimes C \rightarrow \mathbb{k}$  satisfying  $\sum \chi(c_1 \otimes c'_1) f(c_2) f(c'_2) = \sum f(c_1 c'_1) \chi(c_2 \otimes c'_2)$ . The invertibility of  $\Theta$  is equivalent to the invertibility of  $\chi$ .

Similarly, an isomorphism  $\Sigma : \mathbb{k} \rightarrow \mathbb{k} \square_C f_+$  is just an invertible scalar  $\rho$  such that  $\rho f(1) = \rho 1$ . The axioms of a monoidal structure for  $\Theta, \Sigma$  translate to the axioms of a monoidal structure on  $f$  for  $\chi, \rho$ .

The relationship between the structures (1) and (3) is as follows. A monoidal structure  $\chi : C \otimes C \rightarrow \mathbb{k}, \rho \in \mathbb{k}$  on  $f$  induces a monoidal structure on  $(f^+ \square_C -)$  given by the  $D$ -comodule morphism  $m \otimes n \mapsto \sum \chi^{-1}(m_{-1} \otimes n_{-1}) m_0 \otimes n_0 : (f^+ \square_C M) \otimes (f^+ \square_C N) \rightarrow f^+ \square_C (M \otimes N)$  and by  $\lambda \mapsto \rho^{-1} \lambda : \mathbb{k} \rightarrow f^+ \square_C \mathbb{k}$ . The proof of the converse, i.e. that when  $C$  is finite-dimensional every monoidal structure on  $(f^+ \square_C -)$  is of this form for a unique  $(\chi, \rho)$  is similar to the one presented above for the case of right comodules.  $\square$

**Corollary 1.** *In the situation above the functors  $u^+ \square_{\mathbb{k}} - = {}_0(-) : \mathcal{M}^D \rightarrow {}^C \mathcal{M}^D$ ,  $(-)_0 = -\square_{\mathbb{k}} u_+ : {}^C \mathcal{M} \rightarrow {}^C \mathcal{M}^D$ , are monoidal.*

*Proof.* It follows immediately from the fact that  $u : \mathbb{k} \rightarrow C$  preserves the product, the unit and the associator. Explicitly,  $u$  has a monoidal structure provided by  $\chi = \text{id} : \mathbb{k} \rightarrow \mathbb{k}$  and  $\rho = 1 \in \mathbb{k}$ . In this case, equalities (6) and (8) are trivial while condition (7) reads as  $\phi(1 \otimes 1 \otimes 1) = 1$ , which is true by (5).  $\square$

**2.4. Some useful monoidal functors on comodule categories.** In this subsection we describe the functors we use along this work.

**Definition 7.** If  $C$  and  $D$  are coalgebras we define the functors

$$\begin{aligned} (-)^\circ : {}^C\mathcal{M}^D &\rightarrow {}^{D^{\text{cop}}}\mathcal{M}^{C^{\text{cop}}} & (-)^r : (\mathcal{M}_f^C)^{\text{op}} &\rightarrow {}^C\mathcal{M}_f \\ & & (-)^\ell : ({}^C\mathcal{M}_f)^{\text{op}} &\rightarrow \mathcal{M}_f^C \end{aligned}$$

The functor  $(-)^\circ$  is the identity on arrows, and if  $M \in {}^C\mathcal{M}^D$  with coaction  $\chi(m) = \sum m_{-1} \otimes m_0 \otimes m_1$ , then  $M^\circ$  has  $M$  as underlying space and coaction  $\chi^\circ(m) = \sum m_1 \otimes m_0 \otimes m_{-1}$ . In the case when  $C, D$  are coquasi bialgebras,  $(-)^\circ$  has a canonical structure of a monoidal functor  $({}^C\mathcal{M}^D)^{\text{rev}} \rightarrow {}^{D^\circ}\mathcal{M}^{C^\circ}$  given by the usual symmetry of vector spaces  $\text{sw} : M^\circ \otimes N^\circ \cong (N \otimes M)^\circ$  and the identity  $\mathbb{k} \rightarrow \mathbb{k}^\circ$ .

The functor  $(-)^r$  is defined as follows. If  $M \in \mathcal{M}_f^C$ , the underlying space of  $M^r$  is  $M^\vee$ , the linear dual of  $M$ . If  $c$  and  $e$  denote the standard coevaluation and evaluation, the coaction for  $M^r$  is:

$$M^\vee \xrightarrow{\text{id} \otimes c} M^\vee \otimes M \otimes M^\vee \xrightarrow{\text{id} \otimes \chi \otimes \text{id}} M^\vee \otimes M \otimes C \otimes M^\vee \xrightarrow{e \otimes \text{id} \otimes \text{id}} C \otimes M^\vee. \quad (17)$$

On arrows,  $(-)^r$  is given by the usual (linear) duality functor. We call  $M^r$  the *right adjoint* of  $M$ . When  $C$  is a coquasi bialgebra,  $(-)^r$  has the following canonical structure of a monoidal functor  $(\mathcal{M}_f^C)^{\text{op}} \rightarrow {}^C\mathcal{M}_f$ . The unit constraint is the canonical isomorphism  $\mathbb{k} \cong \mathbb{k}^\vee$ ; if  $M, N \in \mathcal{M}_f^C$ , then the transformation  $M^r \otimes N^r \rightarrow (M \otimes N)^r$  is given by the canonical arrows  $M^\vee \otimes N^\vee \rightarrow (M \otimes N)^\vee$ , which are isomorphisms by dimension considerations. We should remark that here we are not thinking  $M^\vee$  as a categorical dual of the vector space  $M$  but rather as the internal hom  $\mathbf{Vect}(M, \mathbb{k})$ . This is the reason why  $(-)^r$  does not reverse the order of the tensor products.

The definition of  $(-)^\ell$  is analogous, if  $N \in {}^C\mathcal{M}_f$ , then:

$$N^\vee \xrightarrow{c \otimes \text{id}} N^\vee \otimes N \otimes N^\vee \xrightarrow{\text{id} \otimes \chi \otimes \text{id}} N^\vee \otimes C \otimes N \otimes N^\vee \xrightarrow{\text{id} \otimes \text{id} \otimes e} C \otimes N^\vee. \quad (18)$$

If  $N \in {}^C\mathcal{M}_f$ , we call  $N^\ell$  the *left adjoint* of  $N$ . When  $C$  is a coquasi bialgebra we have a monoidal functor  $(-)^\ell : ({}^C\mathcal{M}_f)^{\text{op}} \rightarrow \mathcal{M}_f^C$ .

For future reference we record the following results that can be proved directly.

**Lemma 1.** *Observe that  $(-)^r$  and  $(-)^\ell$  are inverse monoidal equivalences and that  $(-)^{r\ell} = (-)^{\ell r}$ . The monoidal isomorphisms  $M^{r\ell} \cong M \cong M^{\ell r}$  are just the canonical linear isomorphisms  $M \cong M^{\vee\vee}$ .*

**Lemma 2.** *For any morphism of coalgebras  $f : C \rightarrow D$ , the diagrams in Figure 2 commute. If moreover  $f$  is a monoidal morphism, the diagrams commute as diagrams of monoidal functors.*

*Proof.* Recall that if  $f$  has a monoidal structure  $\chi : C \otimes C \rightarrow \mathbb{k}, \rho \in \mathbb{k}$ , the monoidal structures on  $(-\square_C f_+)$  and  $(f^+\square_C -)$  are induced by  $\chi, \rho$  and  $\chi^{-1}, \rho^{-1}$ , respectively. Also, it is easy to show that the monoidal structures on  $(-\square_C f_+)^{\text{op}}$  and  $(f^+\square_C -)^{\text{op}}$  are induced by  $\chi^{-1}, \rho^{-1}$  and  $\chi, \rho$  respectively. The verification of the Lemma is direct.  $\square$

$$\begin{array}{ccc}
(\mathcal{M}_f^C)^{\text{op}} & \xrightarrow{(-\square_C f_+)^{\text{op}}} & (\mathcal{M}_f^D)^{\text{op}} \\
(-)^r \downarrow & & \downarrow (-)^r \\
{}^C\mathcal{M}_f & \xrightarrow{f_+ \square_C -} & {}^D\mathcal{M}_f
\end{array}
\quad
\begin{array}{ccc}
({}^C\mathcal{M}_f)^{\text{op}} & \xrightarrow{(f_+ \square_C -)^{\text{op}}} & ({}^D\mathcal{M}_f)^{\text{op}} \\
(-)^{\ell} \downarrow & & \downarrow (-)^{\ell} \\
\mathcal{M}_f^C & \xrightarrow{-\square_C f_+} & \mathcal{M}_f^D
\end{array}$$
  

$$\begin{array}{ccc}
(\mathcal{M}^C)^{\text{rev}} & \xrightarrow{(-\square_D f_+)^{\text{rev}}} & (\mathcal{M}^D)^{\text{rev}} \\
(-)^{\circ} \downarrow & & \downarrow (-)^{\circ} \\
{}^{C^{\circ}}\mathcal{M} & \xrightarrow{f^{\text{cop}} + \square_{C^{\text{cop}}} -} & {}^{D^{\circ}}\mathcal{M}
\end{array}$$

FIGURE 1. Diagrams in Lemma 2

**Lemma 3.** *For any coquasi bialgebra  $C$  the following diagram of monoidal functors commutes.*

$$\begin{array}{ccc}
({}^C\mathcal{M}_f)^{\text{oprev}} & \xrightarrow{((-)^{\circ})^{\text{op}}} & (\mathcal{M}_f^{C^{\circ}})^{\text{op}} \\
((-)^{\ell})^{\text{rev}} \downarrow & & \downarrow (-)^r \\
(\mathcal{M}_f^C)^{\text{rev}} & \xrightarrow{(-)^{\circ}} & {}^{C^{\circ}}\mathcal{M}_f
\end{array}$$

**Lemma 4.** *The following two monoidal functors are monoidally isomorphic to the identity functor via the canonical maps  $M \mapsto M^{\vee\vee}$*

$$\begin{aligned}
& \mathcal{M}_f^C \xrightarrow{(-)^r} {}^C\mathcal{M}_f^{\text{op}} \xrightarrow{(-)^{\circ}} (\mathcal{M}_f^{C^{\circ}})^{\text{oprev}} \xrightarrow{(-)^r} {}^{C^{\circ}}\mathcal{M}_f^{\text{rev}} \xrightarrow{(-)^{\circ}} \mathcal{M}_f^C \\
& {}^C\mathcal{M}_f \xrightarrow{(-)^{\ell}} (\mathcal{M}_f^C)^{\text{op}} \xrightarrow{(-)^{\circ}} ({}^{C^{\circ}}\mathcal{M}_f)^{\text{oprev}} \xrightarrow{(-)^{\ell}} (\mathcal{M}_f^{C^{\circ}})^{\text{rev}} \xrightarrow{(-)^{\circ}} \mathcal{M}_f^C
\end{aligned}$$

### 3. DUALITY

In the case that the map  $S$  is invertible –for example if the coquasi Hopf algebra  $H$  is finite dimensional– the monoidal categories  ${}^H\mathcal{M}_f$ ,  $\mathcal{M}_f^H$  and  ${}^H\mathcal{M}_f^H$  are rigid. In the case of  ${}^H\mathcal{M}^H$ , *e.g.*, we need to construct for every object  $M$  a left and a right dual –denoted as  ${}^*M$  and  $M^*$  respectively– together with the corresponding evaluation and coevaluation maps.

If  $(M, \chi) \in {}^H\mathcal{M}_f^H$  and  $M^{\vee}$  is the dual of the underlying vector space,  $M^* = (M^{\vee}, \chi^*)$  where  $\chi^*$  is the composition

$$\begin{aligned}
M^{\vee} & \xrightarrow{c \otimes \text{id}} M^{\vee} \otimes M \otimes M^{\vee} \xrightarrow{\text{id} \otimes \chi \otimes \text{id}} M^{\vee} \otimes H \otimes M \otimes H \otimes M^{\vee} \xrightarrow{\text{sw} \otimes \text{id} \otimes \text{sw}} \\
H \otimes M^{\vee} \otimes M \otimes M^{\vee} \otimes H & \xrightarrow{\text{id} \otimes \text{id} \otimes e \otimes \text{id}} H \otimes M^{\vee} \otimes H \xrightarrow{S \otimes \text{id} \otimes \bar{S}} H \otimes M^{\vee} \otimes H.
\end{aligned} \tag{19}$$

The evaluation and coevaluation morphisms are given by

$$\text{ev}^{\ell} : {}^*M \otimes M \xrightarrow{\text{id} \otimes \chi} {}^*M \otimes H \otimes M \otimes H \xrightarrow{\text{id} \otimes \beta \bar{S} \otimes \text{id} \otimes \alpha} {}^*M \otimes M \xrightarrow{c} \mathbb{k} \tag{20}$$

and

$$\text{coev}^{\ell} : \mathbb{k} \xrightarrow{c} M \otimes {}^*M \xrightarrow{\chi \otimes \text{id}} H \otimes M \otimes H \otimes {}^*M \xrightarrow{\alpha \bar{S} \otimes \text{id} \otimes \beta \otimes \text{id}} M \otimes {}^*M \tag{21}$$

It is not hard to check using (9) that  $\text{ev}^\ell$  and  $\text{coev}^\ell$  are morphisms in  ${}^H\mathcal{M}^H$ . Moreover, the maps  $\text{ev}^\ell : {}^*M \otimes M \rightarrow \mathbb{k} \in {}^H\mathcal{M}^H$  and  $\text{coev}^\ell : \mathbb{k} \rightarrow M \otimes {}^*M \in {}^H\mathcal{M}^H$  satisfy

$$\text{id}_M = M \xrightarrow{\text{coev}^\ell \otimes \text{id}} (M \otimes {}^*M) \otimes M \xrightarrow{\Phi_{M, {}^*M, M}} M \otimes ({}^*M \otimes M) \xrightarrow{\text{id} \otimes \text{ev}^\ell} M \quad (22)$$

and

$$\text{id}_{{}^*M} = {}^*M \xrightarrow{\text{id} \otimes \text{coev}^\ell} {}^*M \otimes (M \otimes {}^*M) \xrightarrow{\Phi_{{}^*M, M, {}^*M}^{-1}} ({}^*M \otimes M) \otimes {}^*M \xrightarrow{\text{ev}^\ell \otimes \text{id}} {}^*M. \quad (23)$$

These equations are direct consequences of (10) and (11).

Analogously,  ${}^*M = (M^\vee, {}^*\chi)$ , where  ${}^*\chi$  is the composition

$$\begin{aligned} M^\vee &\xrightarrow{\text{id} \otimes c} M^\vee \otimes M \otimes M^\vee \xrightarrow{\text{id} \otimes \chi \otimes \text{id}} M^\vee \otimes H \otimes M \otimes H \otimes M^\vee \xrightarrow{\text{sw} \otimes \text{id} \otimes \text{sw}} \\ H \otimes M^\vee \otimes M \otimes M^\vee \otimes H &\xrightarrow{\text{id} \otimes e \otimes \text{id} \otimes \text{id}} H \otimes M^\vee \otimes H \xrightarrow{\bar{S} \otimes \text{id} \otimes S} H \otimes M^\vee \otimes H. \end{aligned} \quad (24)$$

The corresponding right evaluation and coevaluation morphisms are:

$$\text{ev}^r : M \otimes M^* \xrightarrow{\chi \otimes \text{id}} H \otimes M \otimes H \otimes M^* \xrightarrow{\beta \otimes \text{id} \otimes \alpha \otimes \bar{S} \otimes \text{id}} M \otimes M^* \xrightarrow{e} \mathbb{k} \quad (25)$$

and

$$\text{coev}^r : \mathbb{k} \xrightarrow{c} M^* \otimes M \xrightarrow{\text{id} \otimes \chi} M^* \otimes H \otimes M \otimes H \xrightarrow{\text{id} \otimes \alpha \otimes \text{id} \otimes \beta \otimes \bar{S}} M \otimes M^* \quad (26)$$

As before one easily verifies that  $\text{ev}^r$  and  $\text{coev}^r$  are morphisms in  ${}^H\mathcal{M}^H$  and also that they define a right duality.

**Observation 11.** In explicit terms the comodule structures for the duals are given by the following formulæ. If  $(M, \chi) \in {}^H\mathcal{M}^H$  and  $f \in {}^*M$  and  $m \in M$ , then  ${}^*\chi(f) = \sum f_{-1} \otimes f_0 \otimes f_1 \in H \otimes {}^*M \otimes H$  if and only if:

$$\sum f_0(m) f_{-1} \otimes f_1 = \sum f(m_0) \bar{S}(m_{-1}) \otimes S(m_1).$$

Similarly if  $(M, \chi) \in {}^H\mathcal{M}^H$  and  $f \in M^*$  and  $m \in M$ , then  $\chi^*(f) = \sum f_{-1} \otimes f_0 \otimes f_1 \in H \otimes M^* \otimes H$  if and only if:

$$\sum f_0(m) f_{-1} \otimes f_1 = \sum f(m_0) S(m_{-1}) \otimes \bar{S}(m_1).$$

**Lemma 5.** For the category  $\mathcal{M}^H$ , the duality functors can be expressed in terms of the functors in Definition 7, in the following way.

$${}^*(-) : (\mathcal{M}_f^H)^{\text{op}} \xrightarrow{(-)^r} {}^H\mathcal{M} \xrightarrow{(-)^\circ} \mathcal{M}_f^{H^{\text{cop}}} \xrightarrow{-\square_{H^{\text{cop}}} S_+} \mathcal{M}_f^H$$

$$(-)^* : (\mathcal{M}_f^H)^{\text{op}} \xrightarrow{((-)^\circ)^{\text{op}}} ({}^H\mathcal{M}_f^{\text{cop}})^{\text{op}} \xrightarrow{(-)^\ell} \mathcal{M}_f^{H^{\text{cop}}} \xrightarrow{-\square_{H^{\text{cop}}} \bar{S}_+} \mathcal{M}_f^H$$

**Theorem 3.** If  $H$  is a finite dimensional coquasi Hopf algebra, then its antipode has a canonical structure of a monoidal morphisms of coquasi bialgebras  $S : H^\circ \rightarrow H$ . Moreover, this structure is given by the functional  $\chi^S : H \otimes H \rightarrow \mathbb{k}$

$$\begin{aligned} \chi^S(x \otimes y) &= \sum \phi^{-1}(S(y_3) \otimes S(x_3) \otimes x_5) \alpha(x_4) \phi(S(y_2) S(x_2) \otimes x_6 \otimes y_5) \\ &\quad \alpha(y_4) \beta(x_8 y_7) \phi(S(y_1) S(x_1) \otimes (x_7 y_6) \otimes S(x_9 y_8)). \end{aligned}$$

and corresponds to the usual monoidal structure of the left dual functor  ${}^*(-)$ .

*Proof.* By general categorical principles, the left dual functor has a canonical monoidal structure  $*(-) : (\mathcal{M}^H)^{\text{oprev}} \rightarrow \mathcal{M}^H$ . This can be explicitly computed in terms of the coquasi Hopf algebra structure of  $H$ . On the other hand, we know the monoidal structures of the equivalences  $(-)^r$  and  $(-)^\circ$ , hence we can explicitly compute the monoidal structure of  $(-\square_{H^{\text{cop}}} S_+)$ . The latter is given by a monoidal structure on the coquasi bialgebra morphism  $S : H^\circ \rightarrow H$  (see Theorem 2), and in fact it is given by the functional  $\chi^S$  above.  $\square$

Note that the formula for  $\chi^S$  above is written in function of the comultiplication of  $H$  not of the domain of  $S$ :  $H^{\text{cop}}$ .

The functional in the theorem above appeared in [3], and in the dual case of quasi Hopf algebras in [7].

**Theorem 4.** *Let  $H$  be a finite-dimensional coquasi Hopf algebra and consider the monoidal structure on  $S$  introduced in Proposition 3 above. The canonical linear isomorphisms  $M \cong M^{\vee\vee}$  provide the components for monoidal natural isomorphisms*

$$^{**}(-) \cong -\square_H S_+^2 : \mathcal{M}_f^H \rightarrow \mathcal{M}_f^H \quad (-)^{**} \cong -\square_H \bar{S}_+^2 : \mathcal{M}_f^H \rightarrow \mathcal{M}_f^H.$$

*Proof.* It follows directly from Proposition 3, Lemma 2 applied to  $S : H^{\text{cop}} \rightarrow H$  and Lemma 4.  $\square$

#### 4. THE FUNDAMENTAL THEOREM OF HOPF MODULES

In this section we present a generalization to the case of coquasi-Hopf algebras of the fundamental theorem on Hopf modules introduced by Sweedler in [25]. For a modern presentation of this general case see also [21, 22].

**4.1. The fundamental theorem.** In this section we establish the basic set up in order to state and prove the fundamental theorem on Hopf modules in our context.

In this section, in order to use (formal) duality arguments, we work with a braided monoidal category that we call  $\mathcal{V}$  and with unit object  $\mathbb{k}$  (see [12] as a general reference). Moreover we assume that  $\mathcal{V}$  has equalizers and that the tensor product preserves equalizers in each variable.

The braiding  $\gamma$  is a natural isomorphism  $\gamma_{X,Y} : X \otimes Y \rightarrow Y \otimes X$  and its existence ensures that the natural transformations:

$$\begin{aligned} (X \otimes Y) \otimes (Z \otimes W) &\xrightarrow{\cong} (X \otimes (Y \otimes Z)) \otimes W \xrightarrow{(\text{id} \otimes \gamma_{Y,Z}) \otimes \text{id}} (X \otimes (Z \otimes Y)) \otimes W \\ &\xrightarrow{\cong} (X \otimes Z) \otimes (Y \otimes W) \end{aligned}$$

define a monoidal structure on the functor  $\otimes : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ . Here we are endowing the category  $\mathcal{V} \times \mathcal{V}$  with its usual monoidal structure:  $(X, Y) \otimes (X', Y') = (X \otimes X', Y \otimes Y')$ .

Using the coherence results in [12], we may assume without loss of generality that  $\mathcal{V}$  is a strict monoidal category.

In the above set up, given a braided monoidal category  $\mathcal{V}$ , one can define coalgebra, comodule, coquasi bialgebra and coquasi Hopf algebra objects in  $\mathcal{V}$ . The braiding ensures that the tensor product of two coalgebras is a coalgebra, and likewise with comodules. The fact that the tensor product in  $\mathcal{V}$  preserves

equalizers in each variable, allows us to define the cotensor product of bicomodules in exactly the same manner than in the case that  $\mathcal{V}$  is the category of vector spaces.

If  $C$  is a coquasi bialgebra we denote as  $\mathcal{V}^C$ ,  ${}^C\mathcal{V}$ ,  ${}^C\mathcal{V}^C$ ,  ${}^C\mathcal{V}_C^C$  and  ${}^C\mathcal{V}_C^C$  the categories of right, left and bicomodules, and the category of left and right Hopf modules in  $\mathcal{V}$  respectively. The first three categories have monoidal structures induced by the tensor product of  $\mathcal{V}$ .

The category of bicomodules has also a monoidal structure given by the cotensor product  $\square_C$ .

In the same manner than in Definition 4, we can define the bicomodules  $f^+$  and  $f_+$  for a morphism  $f : C \rightarrow D$  of coalgebra objects in  $\mathcal{V}$ .

For a coquasi bialgebra  $C$  in  $\mathcal{V}$ , we call  $u : \mathbb{k} \rightarrow C$  the unit morphism. In this situation we can consider a pair of adjoint functors  $(u^+ \square_{\mathbb{k}} -) \dashv (u_+ \square_C -) : {}^C\mathcal{V}^C \rightarrow \mathcal{V}^C$ .

Explicitly, if  $M$  is a bicomodule,  $u_+ \square_C M$  is the right comodule of left coinvariants  ${}^{\text{co}C}M$  and if  $N$  is a right comodule,  $u^+ \square_{\mathbb{k}} N$  is the basic object  $N \in \mathcal{V}$  with the same right coaction than  $N$  and with left coaction given by  $u \otimes \text{id}_N : N \rightarrow C \otimes N$ .

**Definition 8.** If  $C$  is a coquasi bialgebra in  $\mathcal{V}$ , we define the free module functor  $F : {}^C\mathcal{V}^C \rightarrow {}^C\mathcal{V}^C$  as  $F(M) = C \otimes M$  for  $M \in {}^C\mathcal{V}^C$ . This functor, together with the forgetful functor  $U : {}^C\mathcal{V}^C \rightarrow {}^C\mathcal{V}^C$  constitute a pair of adjoint functors:  $F \dashv U : {}^C\mathcal{V}^C \rightarrow {}^C\mathcal{V}^C$ . Define the functor  $L : \mathcal{V}^C \rightarrow {}^C\mathcal{V}^C$  as the composition  $L = F \circ (u^+ \square_{\mathbb{k}} -)$ . Clearly  $L$  will have a right adjoint given as  $(u_+ \square_C -) \circ U$ .

The monoidal structure  $\square_C$ , lifts from  ${}^C\mathcal{V}^C$  to the category of Hopf modules in such a way that if  $U : {}^C\mathcal{V}^C \rightarrow {}^C\mathcal{V}^C$  is the forgetful functor, then the adjunction  $F \dashv U : {}^C\mathcal{V}^C \rightarrow {}^C\mathcal{V}^C$  is monoidal (i.e.,  $U$  is lax monoidal,  $F$  is strong monoidal and the unit and counit of the adjunction are monoidal natural transformations).

We want to show that under certain additional hypothesis, the functor  $L$  is a monoidal equivalence.

Without imposing further restrictions on the category  $\mathcal{V}$ , it is not hard to prove the following result, that is analogous to [21, Prop. 3.6] where the difference being that in the above mentioned paper the result is formulated for the case that  $\mathcal{V} = \mathbf{Vect}$  (see also [21, Lemma 2.1]). A general theorem that yields in particular the lemma we present below, appears in [18, Prop. 3.4].

In this lemma we will need the following piece of notation. If  $C, D, C', D'$  are coalgebras in  $\mathcal{V}$  and  $U \in {}^C\mathcal{V}^D$  and  $V \in {}^{C'}\mathcal{V}^{D'}$ , we will denote by  $U \bullet V \in {}^{C \otimes C'}\mathcal{V}^{D \otimes D'}$  the object  $U \otimes V$  equipped with the obvious bicomodule structure.

**Lemma 6.** *The functor  $L : (\mathcal{V}^C, \mathbb{k}, \otimes) \rightarrow ({}^C\mathcal{V}^C, C, \square_C)$  is fully faithful and monoidal.*

*Proof.* It is well known that the functor  $L$  is fully faithful if and only if the unit of the adjunction  $L \dashv (u_+ \square_C -)U$  is an isomorphism. It follows from the dual of [11, Lemma A1.1.1], that it is enough to exhibit a natural isomorphism between  $(u_+ \square_C -)UL$  and the identity functor of  $\mathcal{V}^C$ .

The composition  $UL : \mathcal{V}^C \rightarrow {}^C\mathcal{V}^C$  can be written as  $UL(M) = (C \bullet M) \square_{C \otimes 2p_+}$ . We have natural isomorphisms

$$u_+ \square_C (C \bullet M) \square_{C \otimes 2p_+} \cong (u_+ \bullet M) \square_{C \otimes 2p_+} \cong M \square_C (u_+ \bullet C) \square_{C \otimes 2p_+} \cong M$$

where the last isomorphism is induced by  $(u_+ \bullet C) \square_{C \otimes 2p_+} \cong ((u \otimes \text{id}_C)p)_+ \cong (\text{id}_C)_+ = C$ . This shows that there is a natural isomorphism  $u_+ \square_C UL(M) \cong M$ .

We will now exhibit a canonical monoidal structure on  $L$ . The basic observation is that, for  $M \in \mathcal{M}^C$ ,  $L(M) = C \otimes (u^+ \square_{\mathbb{k}} M)$  is isomorphic to  $(C \bullet M) \square_{C \otimes^2 p_+}$ , where  $C \in {}^C \mathcal{M}^C$  is the regular bicomodule. Then, we can form the composition

$$\begin{aligned}
L(M) \square_C L(N) &\cong ((C \bullet M) \square_{C \otimes^2 p_+}) \square_C ((C \bullet N) \square_{C \otimes^2 p_+}) \\
&\cong (((C \bullet M) \square_{C \otimes^2 p_+}) \bullet N) \square_{C \otimes^2 p_+} \\
&\cong (C \bullet M \bullet N) \square_{C \otimes^3 (p_+ \bullet C)} \square_{C \otimes^2 p_+} & (27) \\
&\cong (C \bullet M \bullet N) \square_{C \otimes^3 (C \bullet p_+)} \square_{C \otimes^2 p_+} & (28) \\
&\cong (C \bullet (M \otimes N)) \square_{C \otimes^2 p_+} \\
&\cong L(M \otimes N).
\end{aligned}$$

All the isomorphisms above follow easily from the definition of the contensor product except for the isomorphism between (27) and (28), which is induced by the isomorphism  $(p_+ \bullet C) \square_{C \otimes^2 p_+} \cong ((p \otimes \text{id}_C)p)_+ \cong ((\text{id}_C \otimes p)p)_+ \cong (C \bullet p_+) \square_{C \otimes^2 p_+}$  that is induced by the associator  $\phi$ .

The isomorphism described above together with the obvious isomorphism  $\mathbb{k} \cong L(\mathbb{k})$  provide a monoidal structure for  $L$ . The axioms of a monoidal functor follow easily from the axioms satisfied by the associator  $\phi$ .  $\square$

In the presence of an antipode, one obtains the following strengthening of the above results. This form of the fundamental theorem on Hopf modules for coquasi Hopf algebras is a consequence of [18, Theorem 7.2]. It follows easily by a simple adaptation of the arguments of the Section 11 of the same work.

**Theorem 5.** *For an arbitrary coquasi Hopf algebra in  $\mathcal{V}$ , the associated functor  $L$  is a monoidal equivalence.*

Observe that we do not ask  $\mathcal{V}$  to be abelian or additive. Neither we assume anything about the existence of duals in  $\mathcal{V}$ . The only requirements on  $\mathcal{V}$  are that it is braided monoidal, with equalizers that are preserved by the tensor product.

For a version of Theorem 5 over **Vect**, see for example [22].

In order to apply this theorem to our context, we need its right version that will be deduced below.

**Observation 12.** Consider the braided monoidal category  $\mathcal{V}^{\text{rev}}$ , which has the same underlying category as  $\mathcal{V}$ , the same unit object but the reverse tensor product  $X \otimes^{\text{rev}} Y = Y \otimes X$ , see Section 1. If  $\gamma_{X,Y} : X \otimes Y \rightarrow Y \otimes X$  is the braiding in  $\mathcal{V}$ , then the braiding in  $\mathcal{V}^{\text{rev}}$  is  $\gamma_{X,Y}^{\text{rev}} = \gamma_{Y,X}$ . The symmetry in the definition of coquasi bialgebra implies that if  $(C, p, u, \phi)$  is a coquasi bialgebra in  $\mathcal{V}$ , then  $(C, p, u, \phi)$  is a coquasi bialgebra in  $\mathcal{V}^{\text{rev}}$ . Moreover, if  $(S, \alpha, \beta)$  is an antipode for the coquasi bialgebra  $H$  in  $\mathcal{V}$ , then  $(S, \beta, \alpha)$  is an antipode for  $H$  in  $\mathcal{V}^{\text{rev}}$ .

**Corollary 2.** *Suppose  $H$  is a coquasi Hopf algebra with invertible antipode in  $\mathcal{V}$ . Then the functor  $R : {}^H \mathcal{V} \rightarrow {}^H \mathcal{V}_H^H$  defined as the composition of  $-\square_{\mathbb{k}} u_+ : {}^H \mathcal{V} \rightarrow {}^H \mathcal{V}^H$  with the free right Hopf module functor  ${}^H \mathcal{V}^H \rightarrow {}^H \mathcal{V}_H^H$  is a monoidal equivalence from  ${}^H \mathcal{V}$  to  ${}^H \mathcal{V}_H^H$ .*

*Proof.* Let us denote  $\mathcal{V}^{\text{rev}}$  by  $\mathcal{W}$ . If  $(H, u, p, \phi)$  is a coquasi bialgebra in  $\mathcal{V}$ , as we saw in Observation 12,  $(H, u, p, \phi)$  is a coquasi bialgebra in  $\mathcal{W}$ , and if  $(S, \alpha, \beta)$  is an antipode for  $H$  in  $\mathcal{V}$  then  $(S, \beta, \alpha)$  is an antipode for  $H$  in  $\mathcal{W}$ .

Clearly, as monoidal categories we have that  ${}^H \mathcal{V} = (\mathcal{W}^H)^{\text{rev}}$ ,  ${}^H \mathcal{V}^H = ({}^H \mathcal{W}^H)^{\text{rev}}$  and  ${}^H \mathcal{V}_H^H = ({}^H \mathcal{W}_H^H)^{\text{rev}}$ . The functor  $-\square_{\mathbb{k}} u_+$  corresponds to  $u^+ \square - : \mathcal{W}^H \rightarrow$



${}^H\mathcal{W}^H$  and the free  $H$ -module functor  ${}^H\mathcal{V}^H \rightarrow {}^H\mathcal{V}_H^H$  to the free  $H$ -module functor  ${}^H\mathcal{W}^H \rightarrow {}^H_H\mathcal{W}^H$ . Then  $R$  is just  $L^{\text{rev}}$  for the coquasi Hopf algebra  $H$  in  $\mathcal{W}$ , and hence it is a monoidal equivalence as we wanted to guarantee.  $\square$

**4.2. The case of invertible antipode.** In this section we prove that the composition of the functors  $u^+\square_{\mathbb{k}}- : \mathcal{M}^H \rightarrow {}^H\mathcal{M}^H$  and the free right  $H$ -module functor  $F : {}^H\mathcal{M}^H \rightarrow {}^H\mathcal{M}_H^H$  is a monoidal equivalence when  $H$  has an invertible antipode.

First we deal with the general case of a coquasi bialgebra.

**Lemma 7.** *For a coquasi bialgebra  $C$ , the functor  $F(u^+\square_{\mathbb{k}}-) : \mathcal{M}^C \rightarrow {}^C\mathcal{M}_C^C$  is left adjoint to  $(u_+\square_H-)U : {}^C\mathcal{M}_C^C \rightarrow \mathcal{M}^C$  where  $U : {}^C\mathcal{M}_C^C \rightarrow {}^C\mathcal{M}^C$  is the forgetful functor. Moreover, the counit transformation corresponding to the above adjunction is the following: if  $M \in {}^C\mathcal{M}_C^C$ , then  $\varepsilon_M : {}_0(\text{co}^C M) \otimes C \rightarrow M$  is the map  $\varepsilon_M(m \otimes c) = m \cdot c$ .*

*Proof.* The assertion about the adjunction is clear. If  $\varepsilon'$  and  $\varepsilon''$  are the counits of  $F \dashv U$  and  $(u^+\square_{\mathbb{k}}-) \dashv (u_+\square_C-)$  respectively, then the counit of the composition of these functors is

$$\varepsilon : F(u^+\square_{\mathbb{k}}-)(u_+\square_C-)U \xrightarrow{F\varepsilon''U} FU \xrightarrow{\varepsilon'} \text{id}.$$

For  $M \in {}^C\mathcal{M}^C$ , the transformation  $\varepsilon''_M$  is the inclusion of  ${}_0(\text{co}^C M)$  in  $M$ , while for  $N \in {}^C\mathcal{M}_C^C$ ,  $\varepsilon'_N : N \otimes C \rightarrow N$  is given by the right action of  $C$  on  $N$ . Therefore  $\varepsilon$  is indeed given by the above formula.  $\square$

**Definition 9.** Let  $H$  be a coquasi Hopf algebra. Define a functor  $\mathcal{I} : \mathcal{M}^H \rightarrow {}^H\mathcal{M}$  as the composition  $\mathcal{M}^H \xrightarrow{(-)^\circ} {}^{H^{\text{cop}}}\mathcal{M} \xrightarrow{S^+\square_{H^{\text{cop}}}-} {}^H\mathcal{M}$ . In other words, on objects  $\mathcal{I}(M, \chi) = (M, (S \otimes \text{id})\text{sw}\chi)$ , and on arrows  $\mathcal{I}$  is the identity.

**Corollary 3.** *In the situation above the functor  $\mathcal{I} : \mathcal{M}^{H^{\text{rev}}} \rightarrow {}^H\mathcal{M}$  is monoidal*

*Proof.* It follows immediately from: a) the equality

$$\mathcal{I} = (S^+\square_{H^{\text{cop}}}-) \circ (-)^\circ : \mathcal{M}^H \xrightarrow{(-)^\circ} {}^{H^{\text{cop}}}\mathcal{M} \xrightarrow{S^+\square_{H^{\text{cop}}}-} {}^H\mathcal{M};$$

b) the fact that  $(-)^\circ$  is monoidal –see the comments after Definition 7–; c) Theorem 2.  $\square$

The following natural transformation will be crucial in the proof of Radford's formula.

**Theorem 6.** *Let  $H$  be a coquasi Hopf algebra with invertible antipode. For  $M \in \mathcal{M}^H$  the arrows  $\tau_M : M \otimes H \rightarrow M \otimes H$  defined as  $\tau_M(m \otimes h) = \sum m_0 \phi^{-1}(m_1 \otimes S(m_3) \otimes h_2) \beta(m_2) \otimes S(m_4) h_1$  are the components of a natural transformation between the functors  $F \circ (-)_0 \circ \mathcal{I}$  and  $F \circ {}_0(-) : \mathcal{M}^H \rightarrow {}^H\mathcal{M}_H^H$ . Moreover, the natural transformation  $\tau$  is invertible and its inverse is given for all  $M \in \mathcal{M}^H$  by the formula:  $\tau_M^{-1}(m \otimes h) = \sum \phi(S(m_1) \otimes m_3 \otimes h_1) \alpha(m_2) m_0 \otimes m_4 h_2$ .*

*Proof.* It is convenient for the proof to split the map  $\tau_M$  as follows. First define the map  $\pi_M : (\mathcal{I}M)_0 \rightarrow {}_0M \otimes H$  as  $\pi_M(m) = \sum m_0 \otimes \beta(m_1) S(m_2)$ . An elementary computation shows that

$$\tau_M : (\mathcal{I}M)_0 \otimes H \xrightarrow{\pi_M \otimes \text{id}} ({}_0M \otimes H) \otimes H \xrightarrow{\Phi_{0M, H, H}} {}_0M \otimes (H \otimes H) \xrightarrow{\text{id} \otimes p} {}_0M \otimes H$$

The structure of  $H$ -bicomodule on  $(F \circ (-)_0 \circ \mathcal{I})(M) = M \otimes H$  is  $m \otimes h \mapsto \sum S(m_1)h_1 \otimes m_0 \otimes h_2 \otimes h_3$ , while the structure on  $(F \circ {}_0(-))(M) = M \otimes H$  is  $m \otimes h \mapsto \sum h_1 \otimes m_0 \otimes h_2 \otimes m_1 h_3$ . A direct verification shows that  $\pi_M$  is a morphism of bicomodules. Hence  $\tau_M$  is a composition of morphisms of bicomodules. The compatibility of  $\tau_M$  with the right action of  $H$  is deduced directly from the fact that  $\pi_M \otimes \text{id}$  and  $(\text{id} \otimes p)\Phi_{0M,H,H}$  are morphisms of right  $H$ -modules. The  $H$ -equivariance of the first morphism is obvious, while the equivariance of the second is a consequence of the following general fact –that we apply in the situation that  $\mathcal{C} = {}^H\mathcal{M}^H$  and  $A = H$ –. If  $(A, p, u)$  is an algebra –also called a monoid– in an arbitrary monoidal category  $\mathcal{C}$  (that in accordance with [12] can be assumed to be strict) then, for any object  $X$  the arrow  $\text{id} \otimes p : X \otimes A \otimes A \rightarrow X \otimes A$  is a morphism of right  $A$ -modules.

Finally, the verification of that the maps  $\tau_M$  and  $\tau_M^{-1}$  are indeed inverses to each other is a direct computation.  $\square$

A version of the above lemma for quasi Hopf algebras appears in [22]. Our proof is similar.

The following result is an immediate consequence of Theorem 6 and Corollary 2.

**Corollary 4.** *In the situation above, the functor  $F(u^+ \square_{\mathbb{k}} -) : (\mathcal{M}^H)^{\text{rev}} \rightarrow {}^H\mathcal{M}_H^H$  has a unique monoidal structure such that  $\tau$  is a monoidal natural transformation. In particular, with this structure  $F(u^+ \square_{\mathbb{k}} -)$  is a monoidal equivalence.*

We end the section with the following observation, that will be used in Section 6.2.

**Observation 13.** If  $M, N \in \mathcal{M}^H$  and  $P, Q \in {}^H\mathcal{M}$ , there exist canonical isomorphisms of bicomodules

$$((u^+ \square_{\mathbb{k}} M) \otimes H) \square_H ((u^+ \square_{\mathbb{k}} N) \otimes H) \cong (u^+ \square_{\mathbb{k}} N) \otimes ((u^+ \square_{\mathbb{k}} M) \otimes H)$$

$$((P \square_{\mathbb{k}} u_+) \otimes H) \square_H ((Q \square_{\mathbb{k}} u_+) \otimes H) \cong (P \square_{\mathbb{k}} u_+) \otimes ((Q \square_{\mathbb{k}} u_+) \otimes H)$$

If  $f : (P \square_{\mathbb{k}} u_+) \rightarrow (u^+ \square_{\mathbb{k}} M)$  and  $g : (Q \square_{\mathbb{k}} u_+) \rightarrow (u^+ \square_{\mathbb{k}} N)$  are morphisms of bicomodules, then the following diagram commutes

$$\begin{array}{ccc}
((P \square_{\mathbb{k}} u_+) \otimes H) \square_H ((Q \square_{\mathbb{k}} u_+) \otimes H) & \xrightarrow{\cong} & (P \square_{\mathbb{k}} u_+) \otimes ((Q \square_{\mathbb{k}} u_+) \otimes H) \\
\downarrow f \square_H g & & \downarrow \text{id} \otimes g \\
& & (P \square_{\mathbb{k}} u_+) \otimes ((u^+ \square_{\mathbb{k}} N) \otimes H) \\
& & \downarrow \Phi^{-1} \\
& & ((P \square_{\mathbb{k}} u_+) \otimes (u^+ \square_{\mathbb{k}} N)) \otimes H \\
& & \downarrow \text{sw} \otimes \text{id} \\
& & ((u^+ \square_{\mathbb{k}} N) \otimes (P \square_{\mathbb{k}} u_+)) \otimes H \\
& & \downarrow \text{id} \otimes f \\
((u^+ \square_{\mathbb{k}} M) \otimes H) \square_H ((u^+ \square_{\mathbb{k}} N) \otimes H) & \xrightarrow{\cong} & (u^+ \square_{\mathbb{k}} N) \otimes ((u^+ \square_{\mathbb{k}} M) \otimes H)
\end{array}$$

## 5. THE FROBENIUS ISOMORPHISM AND THE OBJECT OF COINTEGRALS

Suppose that  $H$  is a coquasi Hopf algebra with invertible antipode. If  $M$  is a left  $H$ -module in the category  ${}^H\mathcal{M}_f^H$  its left dual  ${}^*M$ , is an object in  ${}^H\mathcal{M}_H^H$  in a functorial way. If  $a_M : H \otimes M \rightarrow M$  is a  $H$ -module structure of  $M$  in  ${}^H\mathcal{M}^H$ , the corresponding structure  $a_{{}^*M} : {}^*M \otimes H \rightarrow {}^*M$  is given by

$$\begin{aligned} {}^*M \otimes H &\xrightarrow{\text{id} \otimes \text{coev}^\ell} ({}^*M \otimes H) \otimes (M \otimes {}^*M) \xrightarrow{\cong} ({}^*M \otimes (H \otimes M)) \otimes {}^*M \\ &\xrightarrow{(\text{id} \otimes a_M) \otimes \text{id}} ({}^*M \otimes M) \otimes {}^*M \xrightarrow{\text{ev}^\ell \otimes 1} {}^*M. \end{aligned} \quad (29)$$

From now on we assume that  $H$  is a finite dimensional coquasi Hopf algebra. If we take  $H \in {}^H\mathcal{M}_f^H$  as a left  $H$ -module with respect to the regular action, its right dual  ${}^*H$  is canonically an object in  ${}^H\mathcal{M}_H^H$ . An explicit description of the right  $H$ -structure defined above for  ${}^*H$  is the following: if  $f \in {}^*H$  and  $x, y \in H$ ,  $a_{{}^*H}(f \otimes x)(y) = (f \cdot x)(y)$  is equal to

$$\begin{aligned} (f \cdot x)(y) = & \sum \phi^{-1}(\bar{S}(x_5 y_7) x_1 \otimes y_3 \otimes \bar{S}(y_1)) \alpha \bar{S}(y_2) \phi(\bar{S}(x_4 y_6) \otimes x_2 \otimes y_4) \beta \bar{S}(x_3 y_5) \\ & f(x_6 y_8) \phi(S(x_7 y_9) x_{11} \otimes y_{13} \otimes S(y_{15})) \phi^{-1}(S(x_8 y_{10}) \otimes x_{10} \otimes y_{12}) \alpha(x_9 y_{11}) \beta(y_{14}). \end{aligned} \quad (30)$$

It is important to notice that in the above formula –and in the formula for the Frobenius isomorphism– we obtain the expression for  $f \cdot x \in {}^*H$  in terms of the *standard* evaluation of vector spaces  $H^\vee \otimes H \rightarrow \mathbb{k}$ .

**Theorem 7.** *If  $H$  is a finite dimensional coquasi Hopf algebra, then there exists a unique up to isomorphism one dimensional object  $W \in \mathcal{M}^H$  such that there is an isomorphism  ${}_0W \otimes H \cong {}^*H \in {}^H\mathcal{M}_H^H$ . Moreover,  $W$  can be taken as the space of left cointegrals of the Hopf algebra  $H$  and the isomorphism –called the Frobenius isomorphism– is the map  $\mathcal{F}$  given by*

$$\mathcal{F}(\varphi \otimes x) = \varphi \cdot x \quad (31)$$

where the action used is the one defined in (29) and applied to  $M = {}^*H$  in accordance to the formula (30).

*Proof.* The existence and uniqueness of  $W$  follows immediately from the fundamental theorem on Hopf modules –see more specifically Corollary 2–. The characterization of  $W$  as a space of left cointegrals is deduced directly from the explicit description of the inverse functor of  $F_{\circ 0}(-)$  as the composition of the forgetful functor  $U : {}^H\mathcal{M}_H^H \rightarrow \mathcal{M}_H^H$  with the left fixed part functor –see the considerations previous to Lemma 6–. Thus,  $W$  is the space of left coinvariants of  ${}^*H$  with respect to the coaction described in (24). In explicit terms  $W = \{\varphi \in {}^*H : {}^*\chi(\varphi) = 1 \otimes \varphi\}$ . Using the description of  ${}^*\chi$ , appearing in Observation 11, we conclude that  $\varphi \in W$  if and only if for all  $x \in H$ ,  $\varphi(x)1 = \sum \bar{S}(x_1)\varphi(x_2)$ . In other words  $\varphi(x)1 = \sum x_1\varphi(x_2)$  and then  $\varphi \in {}^*H$  is a left cointegral. The description of the counit of the adjunction as the map given by the action –see Lemma 7– will yield the characterization (31).  $\square$

**Observation 14.** In the same manner than in the classical case, from the existence of the isomorphism  $\mathcal{F}$  we conclude that  $W$ , the space of left cointegrals,

is one dimensional. Hence, one can prove the existence of a group like element  $a \in H$  such that  $\sum \varphi(x_1)x_2 = \varphi(x)a$  for all  $\varphi \in W$  and  $x \in H$ . The element  $a \in H$  is called the *modular* element.

**Observation 15.** Using the definition of the modular element  $a$  just presented as well as formula (30) applied to the situation that  $f = \varphi$  is a cointegral, we obtain the following explicit formula for the Frobenius isomorphism.

$$\begin{aligned} \mathcal{F}(\varphi \otimes x)(y) &= \alpha(a)\beta(1) \\ \sum \phi^{-1}(x_1 \otimes y_2) \leftarrow \alpha \otimes \overline{S}y_1 \varphi(x_2 y_3) \phi(x_3 \otimes \beta \rightarrow y_4 \otimes S y_5) \phi(a^{-1} \otimes a \otimes S y_6) \end{aligned} \quad (32)$$

**Lemma 8.** *The coaction  $\chi_W : W \rightarrow W \otimes H$  is of the form  $\chi_W(\varphi) = \varphi \otimes a^{-1}$  for  $a \in H$  as above.*

*Proof.* As  $W$  is one dimensional, the coaction  $\chi_W$  is of the form  $\chi_W(\varphi) = \varphi \otimes b$  for  $b \in H$ . The definition of the right comodule structure on  $W$  (see Observation 11) yields –for  $x \in H$ – the formula:  $\sum \varphi(x_1)S(x_2) = \varphi(x)b$ . It follows then that  $S(a) = a^{-1} = b$  –see Observation 4–.  $\square$

**Observation 16.** a) The comodule  $W$  is isomorphic to  $a^{-1}_+ \in \mathcal{M}^H$ , where  $a^{-1} : \mathbb{k} \rightarrow H$  is the coalgebra morphism induced by the multiplication by  $a^{-1}$ .  
b) Similarly, the coaction in  ${}^*W$  is given as  $\chi_{*W}(t) = t \otimes a$  for  $t \in {}^*W$ . Hence,  ${}^*W$  is isomorphic to  $a_+ \in \mathcal{M}^H$ .

Recall that we abbreviated the functors  $u^+ \square_{\mathbb{k}} -$  and  $-\square_{\mathbb{k}} u_+$  by  ${}_0(-)$  and  $(-)_0$  respectively.

**Observation 17.**  $W \in \mathcal{M}^H$  as well as  ${}_0W \in {}^H\mathcal{M}^H$  are invertible objects in the corresponding monoidal categories. In other words, the functors  $- \otimes W : \mathcal{M}^H \rightarrow \mathcal{M}^H$  and  $- \otimes {}_0W : {}^H\mathcal{M}^H \rightarrow {}^H\mathcal{M}^H$  are equivalences and it is clear that the inverse equivalences are obtained by tensoring with the corresponding duals.

For use in the next section we write down the following definitions.

**Definition 10.** Define the following functors  $c_W^l, c_W^r : {}^H\mathcal{M}^H \rightarrow {}^H\mathcal{M}^H$  as follows:  $c_W^l = ({}_0W \otimes -) \otimes {}_0{}^*W$  and  $c_W^r = {}_0W \otimes (- \otimes {}_0{}^*W)$

**Observation 18.** It is clear that  $c_W^l$  and  $c_W^r$  are monoidal functors that are naturally isomorphic via the natural transformation given by the obvious associator. In the notations of Theorem 7 and using the fact that  $c_W^l, c_W^r$  are monoidal functors, we conclude that  $({}_0W \otimes H) \otimes {}_0{}^*W$  and  $({}_0W \otimes H) \otimes {}_0{}^*W$  are algebras in the category  ${}^H\mathcal{M}^H$ .

## 6. RADFORD'S FORMULA

In this section we use categorical methods to prove Radford's formula expressing  $S^4$  in terms of conjugation with a functional and a group like element. In the second part of this section we prove the monoidality of the functional.

**6.1. Radford's formula.** We use the notations of the last section and assume that  $H$  is a finite dimensional coquasi Hopf algebra. We will take basis elements  $\varphi \in W, t \in {}^*W$  normalized in such a way that  $t(\varphi) = 1$ .

**Lemma 9.** *In the notations of Theorem 7 the isomorphism in  ${}^H\mathcal{M}^H$*

$$\gamma : ({}_0W \otimes H) \otimes {}^*{}_0W \xrightarrow{\mathcal{F} \otimes \text{id}} {}^*H \otimes {}^*{}_0W \cong {}^*({}_0W \otimes H) \xrightarrow{(*\mathcal{F})^{-1}} {}^{**}H$$

*is a morphism of algebras. Moreover, if we define the Nakayama isomorphism  $\mathcal{N} : H \rightarrow {}^{**}H$  by the formula:  $\mathcal{N}(x) = \gamma((\varphi \otimes x) \otimes t)$ , then the commutativity of the diagram below characterizes  $\mathcal{N}$ :*

$$\begin{array}{ccc} H \otimes (W \otimes H) & \xrightarrow{\cong} & (H \otimes W) \otimes H \\ \mathcal{N} \otimes \mathcal{F} \downarrow & & \downarrow \mathcal{F} \text{sw} \otimes \text{id} \\ {}^{**}H \otimes {}^*H & \xrightarrow{\text{ev}_*^\ell} \mathbb{k} \xleftarrow{\text{ev}_H^\ell} & {}^*H \otimes H \end{array}$$

*Proof.* The multiplicativity of  $\gamma$  follows immediately from the fact that  $\mathcal{F}$  is a morphism of  $H$ -modules and from the commutativity of the following diagram that is a direct consequence of the definition of the action  $a_*H : {}^*H \otimes H \rightarrow {}^*H$ —see Definition (29)—.

$$\begin{array}{ccc} ({}^*H \otimes H) \otimes H & \xrightarrow{\cong} & {}^*H \otimes (H \otimes H) \\ a_*H \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes p \\ {}^*H \otimes H & \xrightarrow{\text{ev}_H^\ell} \mathbb{k} \xleftarrow{\text{ev}_H^\ell} & {}^*H \otimes H \end{array}$$

The assertion concerning  $\mathcal{N}$  follows directly from the definitions.  $\square$

**Observation 19.** a) The fact that  $\gamma$  is a morphism of algebras is valid in the following general context. Let  $\mathcal{C}$  be a rigid monoidal category and let  $a, w \in \mathcal{C}$  be respectively an algebra and an arbitrary object. Let  $\mathcal{F} : w \otimes a \rightarrow {}^*a$  be an invertible morphism of  $a$ -modules in  $\mathcal{C}$ , then the object  $(w \otimes a) \otimes {}^*w$  is an algebra in  $\mathcal{C}$  and the map

$$\gamma : (w \otimes a) \otimes {}^*w \xrightarrow{\mathcal{F} \otimes \text{id}} {}^*a \otimes {}^*w \cong {}^*(w \otimes a) \xrightarrow{(*\mathcal{F})^{-1}} {}^{**}a$$

is a morphism of algebras.

b) In the case of ordinary Hopf algebras, the commutativity of the diagram that characterizes  $\mathcal{N}$  after identifying  $H$  with its double dual reads as  $\varphi(y\mathcal{N}(x)) = \varphi(xy)$  that is the usual definition of the Nakayama automorphism.

**Lemma 10.** *In the notations of Theorem 6 and Theorem 7 if  $M$  is an object in  $\mathcal{M}^H$ , then the morphism  $\xi_M : {}^{**}\mathcal{I}(M)_0 \otimes (({}_0W \otimes H) \otimes {}^*{}_0W) \rightarrow {}^{**}{}_0M \otimes (({}_0W \otimes H) \otimes {}^*{}_0W)$  defined by the commutativity of the diagram below is a morphism of right  $({}_0W \otimes H) \otimes {}^*{}_0W$ -modules.*

$$\begin{array}{ccc} {}^{**}(\mathcal{I}(M)_0 \otimes H) & \xrightarrow{{}^{**}\tau_M} & {}^{**}({}_0M \otimes H) \\ \cong \downarrow & & \downarrow \cong \\ {}^{**}\mathcal{I}(M)_0 \otimes {}^{**}H & \xrightarrow{\omega_M} & {}^{**}{}_0M \otimes {}^{**}H \\ \text{id} \otimes \gamma^{-1} \downarrow & & \downarrow \text{id} \otimes \gamma^{-1} \\ {}^{**}\mathcal{I}(M)_0 \otimes (({}_0W \otimes H) \otimes {}^*{}_0W) & \xrightarrow{\xi_M} & {}^{**}{}_0M \otimes (({}_0W \otimes H) \otimes {}^*{}_0W) \end{array} \quad (33)$$

*Proof.* Being  $\tau_M$  a morphism of  $H$ -modules it is clear that  $\omega_M$  is a morphism of  $**H$ -modules. Then, from the fact that  $\gamma$  is an algebra morphism and that all the modules involved are free over the corresponding algebra objects, it follows that  $\xi_M$  is a morphism of  $(W_0 \otimes H) \otimes {}^*W_0$ -modules.  $\square$

**Definition 11.** Define  $\nu_M : **\mathcal{I}(M)_0 \otimes ({}_0W \otimes H) \rightarrow {}^*_0M \otimes ({}_0W \otimes H)$  as the unique morphism such that the diagram below commutes.

$$\begin{array}{ccc} **\mathcal{I}(M)_0 \otimes (({}_0W \otimes H) \otimes {}^*_0W) & \xrightarrow{\xi_M} & {}^*_0M \otimes (({}_0W \otimes H) \otimes {}^*_0W) \\ \cong \downarrow & & \downarrow \cong \\ (**\mathcal{I}(M)_0 \otimes ({}_0W \otimes H)) \otimes {}^*_0W & \xrightarrow{\nu_M \otimes \text{id}_{{}^*_0W}} & ({}^*_0M \otimes ({}_0W \otimes H)) \otimes {}^*_0W \end{array}$$

The existence of  $\nu_M$  and the fact that it is a morphism in  ${}^H\mathcal{M}^H$  follows immediately from the considerations of Observation 17. Moreover from the fact that  $\xi_M$  is a morphism in  ${}^H\mathcal{M}_{({}_0W \otimes H) \otimes {}^*_0W}^H$ , it follows that  $\nu_M$  is a morphism in  ${}^H\mathcal{M}_H^H$ .

The monoidality of  $\mathcal{I}$  gives canonical isomorphisms  $\mathcal{I}(M^{**}) \cong **\mathcal{I}(M)$ . Composing with  $\nu_M$  we get an isomorphism  $\hat{\nu}_M : \mathcal{I}(M^{**})_0 \otimes ({}_0W \otimes H) \rightarrow {}^*_0M \otimes ({}_0W \otimes H)$ .

For  $M \in \mathcal{M}^H$ , consider the following composition of arrows in  ${}^H\mathcal{M}_H^H$ , from  ${}_0W^* \otimes (({}^*_0M \otimes {}_0W) \otimes H)$  to  ${}_0M^{**} \otimes H$ .

$$\begin{aligned} \zeta_M : {}_0W^* \otimes {}^*_0M \otimes {}_0W \otimes H &\xrightarrow{\text{id} \otimes \hat{\nu}_M^{-1}} {}_0W^* \otimes \mathcal{I}(M^{**})_0 \otimes {}_0W \otimes H \rightarrow \\ &\xrightarrow{\text{id} \otimes \text{sw} \otimes \text{id}} {}_0W^* \otimes {}_0W \otimes \mathcal{I}(M^{**})_0 \otimes H \xrightarrow{\text{ev} \otimes \tau_{M^{**}}} {}_0M^{**} \otimes H \end{aligned} \quad (34)$$

Here we omitted the associativity constraints for simplicity. However this does not introduce any ambiguity as long as we know how to associate the domain and codomain, by the coherence theorem for monoidal categories.

The composition  $\zeta_M$  is a morphism in  ${}^H\mathcal{M}_H^H$ . Indeed,  $\hat{\nu}_M$  and  $\tau_{M^{**}}$  are morphisms of Hopf modules; the morphism  $\text{id} \otimes \text{sw} \otimes \text{id}$  is the image under the free Hopf module functor  ${}^H\mathcal{M}^H \rightarrow {}^H\mathcal{M}_H^H$  of the morphism of bicomodules  $\text{id} \otimes \text{sw} : {}_0W^* \otimes \mathcal{I}(M^{**})_0 \otimes {}_0W \rightarrow {}_0W^* \otimes {}_0W \otimes \mathcal{I}(M^{**})_0$ . Observe that  $\text{sw}$  is a morphism of bicomodules because the trivial comodule structures in each tensor factor are added on opposite sides.

**Definition 12.** Denote by  $\mu_M : W^* \otimes (**M \otimes W) \rightarrow M^{**}$  the unique morphism in  $\mathcal{M}^H$  such that  $\mu_M \otimes \text{id}_H = \zeta_M$ .

It is clear that  $\mu$  is a natural isomorphism between the functors  $W^* \otimes (**(-) \otimes W)$  and  $(-)^{**} : \mathcal{M}^H \rightarrow \mathcal{M}^H$ .

**Corollary 5.** The canonical linear isomorphisms  $M \cong M^{\vee\vee}$  together with  $\mu$  give a natural isomorphism in  $\mathcal{M}^H$

$$M \square_H p(a^{-1} \otimes p(\overline{S}^2 \otimes a))_+ \rightarrow M \square_H S_+^2. \quad (35)$$

*Proof.* First we use Theorem 4, and substitute  $**M$  by  $M \square_H S_+^2$  and  $M^{**}$  by  $M \square_H \overline{S}_+^2$ . In this manner we obtain from  $\mu_M$  an isomorphism

$$W^* \otimes ((M \square_H \overline{S}_+^2) \otimes W) \rightarrow M \square_H S_+^2.$$

$$\begin{array}{ccc}
W^* \otimes **M \otimes W \otimes W^* \otimes **N \otimes W & \xrightarrow{\mu_M \otimes \mu_N} & **M \otimes **N \\
\downarrow \text{id} \otimes \text{id} \otimes \text{ev} \otimes \text{id} \otimes \text{id} & \nearrow \mu_{M \otimes N} & \\
W^* \otimes M \otimes N \otimes W & & 
\end{array}
\quad
\begin{array}{ccc}
& & \mathbb{k} \\
& \swarrow & \downarrow \text{id} \\
W^* \otimes \mathbb{k} \otimes W & \xrightarrow{\mu_{\mathbb{k}}} & \mathbb{k}
\end{array}
\quad (37)$$

FIGURE 2.

Now using the fact that  $W \cong a_+^{-1}$  –Observation 16– and the conclusions of Observation 10 part b) we deduce our result.  $\square$

**Theorem 8** (Radford’s formula). *There exists an invertible functional  $\sigma : H \rightarrow \mathbb{k}$  such that for all  $x \in H$*

$$a^{-1}(\bar{S}^2(x)a) = S^2(\sigma \rightharpoonup x \leftharpoonup \sigma^{-1}).$$

*Proof.* It easily follows from Corollary 5 and Theorem 1. Indeed in the situation of Corollary 5 the theorem guarantees the existence of a functional  $\sigma$  such that –see Observation 8, (13)–  $p(a^{-1} \otimes p(\bar{S}^2 \otimes a))(x) = S^2(\sigma \rightharpoonup x \leftharpoonup \sigma^{-1})$ .  $\square$

The functional  $\sigma$  defined in the theorem above is the analogue for finite dimensional coquasi Hopf algebras of the modular function of a finite dimensional Hopf algebra. See Section 7.

**Observation 20.** The above formula can be transformed into another similar to the classical formula:

$$S^4(x) = (a^{-1}(\hat{\sigma} \rightharpoonup x \leftharpoonup \hat{\sigma}^{-1}))a \quad (36)$$

where  $\hat{\sigma}$  is another invertible functional that can be computed explicitly in terms of the above information.

**6.2. Monoidality.** In this section we prove that the natural isomorphism  $\mu$  of Definition 12 is monoidal. We shall work as if the monoidal category  $({}^H\mathcal{M}^H, \mathbb{k}, \otimes)$  were strict, and hence ignore the associativity and unit constraints. This can be formalized by passing to an monoidally equivalent strict monoidal category. Indeed, our proof does not depend on the fact that we are working with the category of comodules, but only on certain properties satisfied by the several arrows we consider.

The functor  $\mathcal{M}^H \rightarrow \mathcal{M}^H$  given by  $M \mapsto W^* \otimes M \otimes W$  has a canonical monoidal structure given by the constraints

$$\begin{aligned}
W^* \otimes M \otimes W \otimes W^* \otimes N \otimes W & \xrightarrow{\text{id} \otimes \text{id} \otimes \text{ev} \otimes \text{id} \otimes \text{id}} W^* \otimes M \otimes N \otimes W \\
\mathbb{k} & \xrightarrow{\text{coev}} W^* \otimes W \xrightarrow{\cong} W^* \otimes \mathbb{k} \otimes W
\end{aligned}$$

These morphisms are isomorphisms because  $W$  is an invertible object –it has dimension one–.

**Theorem 9.** *The natural transformation  $\mu$  in Definition 12 is monoidal.*

The assertion that  $\mu$  is a monoidal natural transformation is expressed in the commutativity of the diagrams in Figure 2.

$$\begin{array}{ccccc}
({}_0W^*)(**_0M)({}_0W)({}_0W^*)(**_0N)({}_0W)H & \xrightarrow{(\text{id})\text{ev}(\text{id})} & ({}_0W^*)(**_0M)({}_0W^*)(**_0N)({}_0W)H & \xrightarrow{\cong} & ({}_0W^*)(**_0(MN))({}_0W)H \\
\downarrow (\text{id})\bar{\nu}_N^{-1} & & \downarrow (\text{id})\bar{\nu}_N^{-1} & & \downarrow (\text{id})\bar{\nu}_{MN}^{-1} \\
({}_0W^*)(**_0M)({}_0W)({}_0W^*)(\mathcal{I}(N^{**}))({}_0W)H & \longrightarrow & ({}_0W^*)(**_0M)(\mathcal{I}(N^{**}))({}_0W)H & & \\
\downarrow (\text{id})\text{sw}(\text{id}) & & \downarrow (\text{id})\text{sw}(\text{id}) & & \downarrow (\text{id})\bar{\nu}_{MN}^{-1} \\
({}_0W^*)(**_0M)({}_0W)({}_0W^*)(\mathcal{I}(N^{**}))H & \longrightarrow & ({}_0W^*)(**_0M)({}_0W)(\mathcal{I}(N^{**})_0)H & \text{(C)} & ({}_0W^*)(\mathcal{I}(MN)_0)({}_0W)H \\
\downarrow (\text{id})(\text{id})(\text{id})\text{ev}(\text{id})(\text{id}) & \nearrow & \downarrow \text{sw}(\text{id}) & & \downarrow (\text{sw})(\text{id})(\text{id}) \\
({}_0W^*)(**_0M)({}_0W)(\mathcal{I}(N^{**})_0)H & & (\mathcal{I}(N^{**})_0)({}_0W^*)(**_0M)({}_0W)H & & \\
\downarrow \text{id}\hat{\tau}_{N^{**}} & & \downarrow \text{id}\bar{\nu}_M^{-1} & & \downarrow (\text{id})\text{ev}(\text{id}) \\
({}_0W)(**_0M)({}_0W)({}_0N^{**})H & & (\mathcal{I}(N^{**})_0)({}_0W^*)(\mathcal{I}(M^{**})_0)({}_0W)H & \longrightarrow & (\mathcal{I}((MN)^{**})_0)({}_0W^*)({}_0W)H \\
\downarrow ({}_0\mu_M)(\text{id})(\text{id}) & & \downarrow (\text{id})(\text{id})\text{ev}(\text{id}) & & \downarrow (\text{id})\text{ev}(\text{id}) \\
({}_0W)(**_0M)({}_0W)({}_0N^{**})H & & (\mathcal{I}(N^{**})_0)(\mathcal{I}(M^{**})_0)({}_0W^*)({}_0W)H & & \\
\downarrow ({}_0\mu_M)(\text{id})(\text{id}) & & \downarrow (\text{id})\tau_{M^{**}} & & \downarrow \tau_{(MN)^{**}} \\
({}_0W)(**_0M)({}_0W)({}_0N^{**})H & & (\mathcal{I}(N^{**})_0)({}_0M^{**})H & & \\
\downarrow ({}_0\mu_M)(\text{id})(\text{id}) & & \downarrow \text{sw}(\text{id}) & \text{(B)} & \\
({}_0W)(**_0M)({}_0W)({}_0N^{**})H & & ({}_0M^{**})(\mathcal{I}(N^{**})_0)H & & \\
\downarrow ({}_0\mu_M)(\text{id})(\text{id}) & \nearrow \text{id}\tau_{N^{**}} & & & \\
({}_0M^{**})({}_0N^{**})H & \xrightarrow{\cong} & & & ({}_0(MN)^{**})H
\end{array}$$

FIGURE 3.

$$\begin{array}{ccccc}
(\mathcal{I}(M)_0 \otimes H) \square_H (\mathcal{I}(N)_0 \otimes H) & \xrightarrow{\cong} & \mathcal{I}(N)_0 \otimes \mathcal{I}(M)_0 \otimes H & \longrightarrow & \mathcal{I}(M \otimes N)_0 \otimes H \\
\downarrow \tau_M \square_H \tau_N & & \downarrow \text{id} \otimes \tau_M & & \downarrow \tau_{M \otimes N} \\
({}_0M \otimes H) \square_H ({}_0N \otimes H) & \xrightarrow{\cong} & {}_0M \otimes {}_0N \otimes H & \longrightarrow & {}_0(M \otimes N) \otimes H \\
\downarrow \tau_M \square_H \tau_N & & \downarrow \text{id} \otimes \tau_N & & \downarrow \tau_{M \otimes N} \\
({}_0M \otimes H) \square_H ({}_0N \otimes H) & \xrightarrow{\cong} & {}_0M \otimes {}_0N \otimes H & \longrightarrow & {}_0(M \otimes N) \otimes H
\end{array}$$

FIGURE 4.



$$\begin{array}{ccc}
{}_0W^* \otimes {}^{**}_0M \otimes {}_0W \otimes \mathcal{I}(N^{**})_0 \otimes H & \xrightarrow{\text{sw} \otimes \text{id}} & \mathcal{I}(N^{**})_0 \otimes {}_0W^* \otimes {}^{**}_0M \otimes {}_0W \otimes H \\
\downarrow \text{id} \otimes \text{id} \otimes \text{id} \otimes \tau_{N^{**}} & \searrow & \downarrow \text{id} \otimes {}_0\mu_M \otimes \text{id} \\
{}_0W^* \otimes {}^{**}_0M \otimes {}_0W \otimes {}_0N^{**} \otimes H & \xrightarrow{{}_0\mu_M \otimes \text{id} \otimes \text{id}} & \mathcal{I}(N^{**})_0 \otimes {}_0M^{**} \otimes H \\
\downarrow {}_0\mu_M \otimes \text{id} \otimes \text{id} & & \downarrow \text{sw} \otimes \text{id} \\
{}_0M^{**} \otimes {}_0N^{**} \otimes H & \xleftarrow{\text{id} \otimes \tau_{N^{**}}} & {}_0M^{**} \otimes \mathcal{I}(N^{**})_0 \otimes H
\end{array}$$

FIGURE 5.

$$\begin{array}{ccc}
{}^{**}_0M \otimes {}^{**}_0N \otimes {}_0W \otimes H & \xrightarrow{\cong} & {}^{**}_0(M \otimes N) \otimes {}_0(M \otimes N) \otimes {}_0W \otimes H \\
\downarrow \text{id} \otimes \nu_N^{-1} & & \downarrow \nu_{M \otimes N}^{-1} \\
{}^{**}_0M \otimes {}^{**}\mathcal{I}(N)_0 \otimes {}_0W \otimes H & & {}^{**}\mathcal{I}((M \otimes N)^{**})_0 \otimes {}_0W \otimes H \\
\downarrow \text{sw} \otimes \text{id} & & \uparrow \cong \\
{}^{**}\mathcal{I}(N)_0 \otimes {}^{**}_0M \otimes {}_0W \otimes H & \xrightarrow{\text{id} \otimes \nu_M^{-1}} & {}^{**}\mathcal{I}(N)_0 \otimes {}^{**}\mathcal{I}(M)_0 \otimes {}_0W \otimes H
\end{array}$$

FIGURE 6.

$$\begin{array}{ccc}
{}^{**}_0M \otimes {}^{**}_0N^{**} H & \xrightarrow{\cong} & {}^* * {}_0(M \otimes N) \otimes {}^{**}H \\
\downarrow \text{id} \otimes {}^{**}\tau_N^{-1} & & \downarrow {}^{**}\tau_{M \otimes N}^{-1} \\
{}^{**}_0M \otimes {}^{**}\mathcal{I}(N)_0 \otimes {}^{**}H & & {}^{**}\mathcal{I}(M \otimes N)_0 \otimes {}^{**}H \\
\downarrow \text{sw} \otimes \text{id} & & \uparrow \cong \\
{}^{**}\mathcal{I}(N)_0 \otimes {}^{**}_0M \otimes {}^* * H & \xrightarrow{\text{id} \otimes {}^{**}\tau_M^{-1}} & {}^{**}\mathcal{I}(N)_0 \otimes {}^{**}\mathcal{I}(M)_0 \otimes {}^{**}H
\end{array}$$

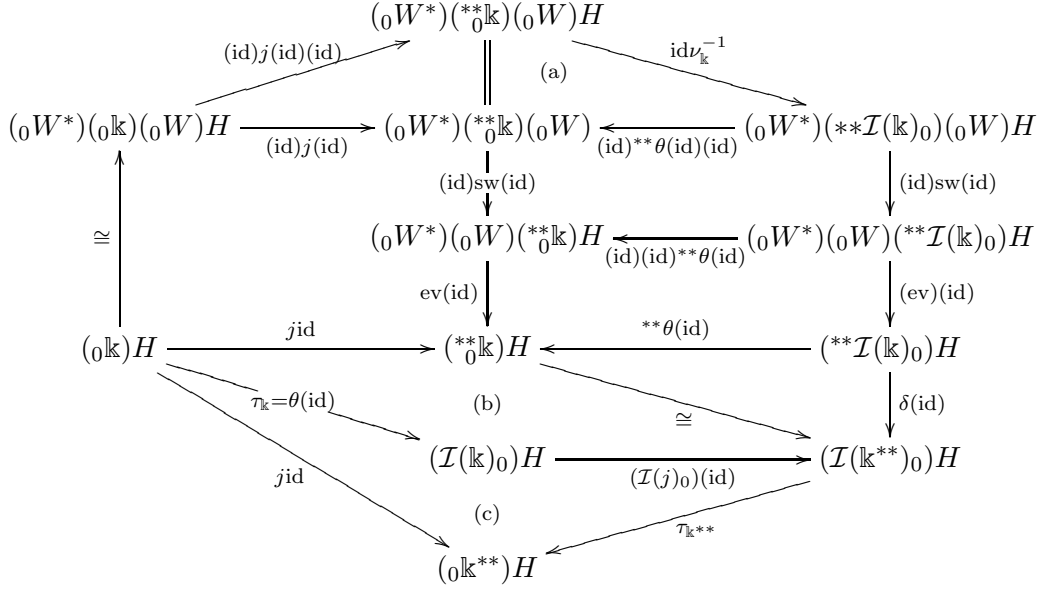
FIGURE 7.

*Proof.* We divide the proof in two parts. In some diagrams, we omit the symbol  $\otimes$  as a saving space measure, adding parenthesis when necessary.

*First axiom.* The image of the diagram on the left hand side of Figure 2 is the exterior rectangle in Figure 3. So it is enough to show the latter commutes, as the functor  $M \mapsto {}_0M \otimes H$  is an equivalence by Corollary 4. The sub diagrams left blank commute trivially.

The diagram marked by (A) is just the commutative rectangle in Figure 5. This is easy to show using the naturality of  $\text{sw}$  and the definition of  $\mu$ . The diagram (B) in Figure 3 commutes if the diagram marked by (E) in Figure 4 commutes for all  $M, N$ . To show this, observe that the exterior rectangle in Figure 4 commutes by monoidality of  $\tau$  and that the sub diagram (D) commutes by Observation 13.

Finally, the diagram marked by (C) in Figure 4 commutes if and only if the diagram in Figure 6 does. If we tensor this diagram with  ${}_0W^*$  on the right, after



Finally, the diagram (c) commutes by naturality of  $\tau$ .

The monoidality of the natural transformation  $\mu$  just proved translates into properties of the functional  $\sigma$  in Theorem 8. We compute the below in an explicit way the monoidal structure of  $\sigma$ .

**Observation 21.** The functional  $\sigma$  induces the natural isomorphism of Corollary 5, which is monoidal since  $\mu$  is. Therefore, if we know the monoidal structures of the morphisms  $p(a^{-1} \otimes p(\bar{S}^2 \otimes a))$  and  $S^2$ , we can deduce the equations satisfied by  $\sigma$ . More explicitly, if these morphisms have monoidal structures  $(\chi_1, \rho_1)$  and  $(\chi_2, \rho_2)$  respectively, then  $\sigma$  satisfies

$$\chi_1 \star \sigma p = (\sigma \otimes \sigma) \star \chi_2 \quad \rho_1 \sigma(1) = \rho_2. \quad (38)$$

The antipode  $S : H^\circ \rightarrow H$  has a monoidal structure  $(\chi^S, 1)$ , where  $\chi^S$  is given explicitly in Proposition 3. By Observation 2 we have that  $S^2$ , which is the composition of  $S^{\text{cop}} : H \rightarrow H^\circ$  and  $S : H^\circ \rightarrow H$ , has  $(\chi^S(S \otimes S) \star (\chi^S)^{-1} \text{sw}, 1)$  as monoidal structure. This is because  $S^{\text{cop}}$  has a monoidal structure  $((\chi^S)^{-1} \text{sw}, 1)$ . The inverse of the antipode  $\bar{S} : H^\circ \rightarrow H$  has a canonical monoidal structure given in terms of  $\chi^S$  by  $((\chi^S)^{-1}(\bar{S} \otimes \bar{S}), 1)$ . Thus,  $\bar{S}^2$ , this is, the composition of  $\bar{S}$  with  $\bar{S}^{\text{cop}} : H^\circ \rightarrow H$ , has a monoidal structure  $(\chi^S(\bar{S}^2 \otimes \bar{S}^2) \text{sw} \star (\chi^S)^{-1}(\bar{S} \otimes \bar{S}), 1)$ .

The morphism  $p(a^{-1} \otimes p(\text{id} \otimes a)) : H \rightarrow H$  is monoidal with a monoidal structure given by  $(\chi_0, 1)$  where  $\chi_0$  is the following product in  $(H \otimes H)^\vee$ .

$$\phi^{-1}(a^{-1} \otimes (-)a \otimes a^{-1}((?)a)) \star \phi((-)a \otimes a^{-1} \otimes (?)a) \star \phi^{-1}(- \otimes a \otimes a^{-1}) \star \phi(- \otimes ? \otimes a)$$

Then, the monoidal structure  $(\chi_1, \rho_1)$  of the composition of  $\bar{S}^2$  with  $p(a^{-1} \otimes p(\text{id} \otimes a))$  is given by  $\chi_1 = \chi_0(\bar{S}^2 \otimes \bar{S}^2) \star \chi^S(\bar{S}^2 \otimes \bar{S}^2) \text{sw} \star (\chi^S)^{-1}(\bar{S} \otimes \bar{S})$  and  $\rho_1 = 1$ . We deduce that  $\sigma$  satisfies  $\sigma(1) = 1$  and

$$\chi_0(\bar{S}^2 \otimes \bar{S}^2) \star \chi^S(\bar{S}^2 \otimes \bar{S}^2) \text{sw} \star (\chi^S)^{-1}(\bar{S} \otimes \bar{S}) \star \sigma p = (\sigma \otimes \sigma) \star \chi^S(S \otimes S) \star (\chi^S)^{-1} \text{sw}. \quad (39)$$

## 7. THE CASE OF A HOPF ALGEBRA

We briefly mention the needed adjustments to the proof above in order to get the classical Radford's formula for  $S^4$ . We assume that  $H$  is a finite dimensional Hopf algebra and define the following functions.

Denote by  $\omega \in H^\vee$ , the modular function of  $H$  that we know it is an algebra homomorphism. It can be defined as the modular element in the Hopf algebra  $H^\vee$  (the linear dual of  $H$ ). In particular if  $i \in H$  is a right integral, the functional  $\omega$  is characterized by the property that for all  $x \in H$  we have that  $xi = \omega(x)i$ .

We will also consider the automorphism of Nakayama  $\mathcal{N}$ , that is characterized by the equation  $\varphi(xy) = \varphi(y\mathcal{N}(x))$  for all  $x, y \in H$  where  $\varphi$  is as before a right cointegral for  $H$ .

Next we show how to obtain an expression of the inverse of Nakayama's automorphism in terms of  $S$  and  $\omega$ . For all  $x, y \in H$ , we have that:

$$\sum \varphi(y_1 x) y_2 = \sum \varphi(y_1 x_1) y_2 \varepsilon(x_2) = \sum \varphi(y_1 x_1) y_2 x_2 S(x_3) = \sum \varphi(y x_1) S(x_2) \quad (40)$$

If we take  $y = i$  in the above equality and assume that  $\varphi(i) = 1$  we obtain that  $S(x) = \sum \varphi(i_1 x) i_2$  or equivalently that  $x = \sum \varphi(i_1 x) \bar{S} i_2$ . Hence, it follows that  $\mathcal{N}(x) = \sum \varphi(i_1 \mathcal{N} x) \bar{S} i_2 = \bar{S}^2(\sum \varphi(x i_1) S i_2)$ . Now, using again the equation

(40) we conclude that

$$\mathcal{N}(x) = \overline{S}^2(\sum \varphi(x_1 i) x_2) = \overline{S}^2(\sum \omega(x_1) x_2) = \overline{S}^2(x \leftarrow \omega).$$

Taking the inverse maps in the above equation we conclude that  $S^2 x \leftarrow \omega^{-1} = \mathcal{N}^{-1} x$ . Then, it follows that  $\varepsilon \mathcal{N}^{-1} = \omega^{-1}$ .

In the case of a Hopf algebra, for  $M \in \mathcal{M}^H$  then  $\tau_M : M \otimes H \rightarrow M \otimes H$  is given by the formula  $\tau_M(m \otimes h) = \sum m_0 \otimes S(m_1)h$ . This is easily obtained from Theorem 6 substituting the associators as well as  $\alpha$  and  $\beta$  by  $\varepsilon$ . The inverse of  $\tau_M$  is given as  $\tau_M^{-1}(m \otimes h) = \sum m_0 \otimes m_1 h$ .

With respect to the properties of duality, the same formulæ (19) and (24) yields the comodule structure on the dual spaces. The evaluation and coevaluation in this case are the same than the usual ones in the category of vector spaces (see formulæ (20), (21), (25), (26)).

If  $M$  is a left  $H$ -module the natural right  $H$ -module structure on the left dual –see (19)–  $a_M : {}^*M \otimes H \rightarrow {}^*M$  is given by

$${}^*M \otimes H \xrightarrow{\text{id} \otimes \text{id} \otimes c} {}^*M \otimes H \otimes M \otimes {}^*M \xrightarrow{\text{id} \otimes a_M \otimes \text{id}} {}^*M \otimes M \otimes {}^*M \xrightarrow{e \otimes \text{id}} {}^*M.$$

For the right dual the formula is similar –see (19). In particular, in the case we consider  $H \in {}^H\mathcal{M}^H$  as a left module with respect to the regular action, the right  $H$ -structure considered above in this situation is simply the following:  $f \leftarrow h \in {}^*H$ ,  $(f \leftarrow h)(x) = f(hx)$ .

The Frobenius map  $\mathcal{F}$ , that was given in Theorem 7, is  $\mathcal{F}(\varphi \otimes h) = \varphi \leftarrow h$  for  $\varphi \in W$  and  $h \in H$ .

In particular as we mentioned in Observation 19 part b), the morphism of algebras  $\gamma$  defined in Lemma 9:

$$\gamma : {}_0W \otimes H \otimes {}^*W \xrightarrow{\mathcal{F} \otimes \text{id}} {}^*H \otimes {}^*W \cong {}^*({}_0W \otimes H) \xrightarrow{(*\mathcal{F})^{-1}} {}^{**}H,$$

is given by the formula:  $\gamma(\varphi \otimes x \otimes t) = \text{ev}(- \otimes \mathcal{N}x)$  where  $\mathcal{N} : H \rightarrow H$  is the Nakayama morphism, that in this case is an algebra automorphism.

The map  $\xi_M$  considered in Lemma 10, can be described explicitly by  $\xi_M(\text{ev}_m \otimes \varphi \otimes h \otimes t) = \sum \text{ev}_{m_0} \otimes \varphi \otimes \mathcal{N}^{-1}(S(m_1))h \otimes t$ . Indeed, from the commutative diagram (33) we deduce that  $\omega_M(\text{ev}_m \otimes \text{ev}_h) = \sum \text{ev}_{m_0} \otimes \text{ev}_{S(m_1)h}$ . To prove the formula for  $\xi_M$  we prove that

$$(\text{id} \otimes \gamma)(\sum \text{ev}_{m_0} \otimes \varphi \otimes \mathcal{N}^{-1}(S(m_1))h \otimes t) = \omega_M(\text{id} \otimes \gamma)(\text{ev}_m \otimes \varphi \otimes h \otimes t).$$

The left hand side of the above equation is:

$$(\text{id} \otimes \gamma)(\sum \text{ev}_{m_0} \otimes \varphi \otimes \mathcal{N}^{-1}(S(m_1))h \otimes t) = \sum \text{ev}_{m_0} \otimes \text{ev}_{S(m_1)\mathcal{N}(h)},$$

while the right hand side can be computed as:

$$\omega_M(\text{id} \otimes \gamma)(\text{ev}_m \otimes \varphi \otimes h \otimes t) = \omega_M(\text{ev}_m \otimes \text{ev}_{\mathcal{N}(h)}) = \sum \text{ev}_{m_0} \otimes \text{ev}_{S(m_1)\mathcal{N}(h)}.$$

Hence the map  $\nu_M : {}^{**}\mathcal{I}(M)_0 \otimes {}_0W \otimes H \rightarrow {}^{**}M \otimes {}_0W \otimes H$  introduced in Definition 11 is given by:  $\nu_M(\text{ev}_m \otimes \varphi \otimes h) = \sum \text{ev}_{m_0} \otimes \varphi \otimes \mathcal{N}^{-1}(S(m_1))h$ . Moreover, the map  $\widehat{\nu}_M$  has exactly the same expression than  $\nu_M$ .

For later use we record the following formula for  $\widehat{\nu}_M^{-1}$  that can be proved by a direct computation:

$$\widehat{\nu}_M^{-1}(\text{ev}_m \otimes \varphi \otimes h) = \sum \text{ev}_{m_0} \otimes \varphi \otimes \mathcal{N}^{-1}(m_1)h. \quad (41)$$

Thus, the morphism  $\zeta_M$  defined in (34) is given as:

$$\zeta_M(t \otimes \text{ev}_m \otimes \varphi \otimes h) = \sum \text{ev}_{m_0} \otimes \bar{S}(m_1) \mathcal{N}^{-1}(m_2) h.$$

Indeed it follows from equation (19) that the right coaction in  $M^{**}$  is  $\chi_{M^{**}}(\text{ev}_m) = \sum \text{ev}_{m_0} \otimes \bar{S}^2(m_1)$ . Thus,

$$\begin{aligned} \zeta_M(t \otimes \text{ev}_m \otimes \varphi \otimes h) &= (\text{ev} \otimes \tau_{M^{**}})(\text{id} \otimes \text{sw} \otimes \text{id})(\text{id} \otimes \hat{\nu}_M^{-1})(t \otimes \text{ev}_m \otimes \varphi \otimes h) = \\ &= \sum (\text{ev} \otimes \tau_{M^{**}})(\text{id} \otimes \text{sw} \otimes \text{id})(t \otimes \text{ev}_{m_0} \otimes \varphi \otimes \mathcal{N}^{-1}(m_1) h) = \\ &= \tau_{M^{**}}(\sum \text{ev}_{m_0} \otimes \mathcal{N}^{-1}(m_1) h) = \sum \text{ev}_{m_0} \otimes \bar{S}(m_1) \mathcal{N}^{-1}(m_2) h. \end{aligned}$$

Then, as  $\zeta_M = \mu_M \otimes \text{id}_H$ , with  $\mu_M : W^* \otimes^{**} M \otimes W \rightarrow M^{**}$ , it is clear that  $\mu_M$  satisfies the following equality:  $\mu_M(t \otimes \text{ev}_m \otimes \varphi) \otimes h = \sum \text{ev}_{m_0} \otimes \bar{S}(m_1) \mathcal{N}^{-1}(m_2) h$ .

If we apply  $\text{id} \otimes \text{id} \otimes \varepsilon$  to the equality above we obtain:

$$\mu_M(t \otimes \text{ev}_m \otimes \varphi) = \sum \text{ev}_{m_0} (\varepsilon \mathcal{N}^{-1})(m_1) = \sum \text{ev}_{m_0} \omega^{-1}(m_1)$$

We have used above the equality  $\varepsilon \mathcal{N}^{-1} = \omega^{-1}$  proved before. Hence we deduce that  $\mu_M(t \otimes \text{ev}_m \otimes \varphi) = \text{ev}_{\omega^{-1} \rightharpoonup m}$ .

Next, we observe that the natural isomorphism constructed in Corollary 5 is simply the map  $m \mapsto (\omega^{-1} \rightharpoonup m)$ . Applying the bijections proved in Theorem 1, we find that the map  $\sigma$  appearing in Radford's formula –Theorem 8– is simply  $\sigma(h) = \varepsilon(\omega^{-1} \rightharpoonup h) = \omega^{-1}(h)$ . Hence, we deduce the classical Radford's formula

$$a^{-1} \bar{S}^2(x) a = \omega^{-1} \rightharpoonup S^2(x) \leftarrow \omega \quad \text{or} \quad S^4(x) = \omega \rightharpoonup a^{-1} x a \leftarrow \omega^{-1}.$$

This shows that the functional  $\sigma$  is indeed the coquasi Hopf algebra analogue of the modular function.

Next we explain how the monoidality of  $\sigma$  proved at the end of the previous section generalizes the multiplicativity of the modular function  $\omega \in H^\vee$ .

Recall that in the case of a Hopf algebra the associativity of the product and the fact that  $S$  is a morphism of algebras are expressed as  $\phi = \varepsilon \otimes \varepsilon \otimes \varepsilon$  and  $\chi^S = \varepsilon \otimes \varepsilon$ . Therefore, the equality (39) simplifies to  $\sigma p = \sigma \otimes \sigma$ , this is,  $\sigma$  is multiplicative. The equality  $\sigma(1) = 1$  was shown for an arbitrary coquasi Hopf algebra. Hence  $\omega = \sigma^{-1}$  is a morphism of algebras.

An important point is that in the proof of the monoidality of  $\sigma$  we did not use integrals (only cointegrals, presented as the comodule  $W$ ). Then, in the case of a finite dimensional Hopf algebra, if we use  $\sigma$  instead of the modular function, we can avoid mentioning integrals altogether in the proof of the classical Radford's formula.

## 8. APPENDIX: CATEGORICAL BACKGROUND

This appendix is an account of some basic results on functors between categories of comodules. These results are not unknown, but the proofs found in the literature are usually *ad hoc*. The unified presentation below is based on density of functors and completions of categories under certain classes of colimits.

We will work with categories enriched in the category of vector spaces over a field  $\mathbb{k}$ , sometimes called  $\mathbb{k}$ -linear categories. Although one has to be careful when dealing with enriched categories, in our case the subtleties of the theory disappear. This is a consequence of the fact that the *underlying set functor*  $\mathbf{Vect}(\mathbb{k}, -) : \mathbf{Vect} \rightarrow \mathbf{Set}$  is conservative (*i.e.*, reflects isomorphisms). We denote by  $[\mathcal{A}, \mathcal{B}]$

the  $\mathbb{k}$ -linear category of  $\mathbb{k}$ -linear functors  $\mathcal{A} \rightarrow \mathcal{B}$  and natural transformations between them.

Recall the notion of dense functor (see [16] for a complete exposition on the subject). Let  $\mathcal{A}$  be a small category. A functor  $K : \mathcal{A} \rightarrow \mathcal{C}$  is dense if the functor  $\tilde{K} : \mathcal{C} \rightarrow [\mathcal{A}^{\text{op}}, \mathbf{Vect}]$ , given by  $C \mapsto \mathcal{C}(K-, C)$ , is fully faithful. In other words,  $K$  is dense if every natural transformation  $\mathcal{C}(K-, C) \Rightarrow \mathcal{C}(K-, D)$  is of the form  $\mathcal{C}(K-, f)$  for a unique  $f : C \rightarrow D$  in  $\mathcal{C}$ . When  $K$  is the inclusion of a full subcategory, say that  $\mathcal{A}$  is dense in  $\mathcal{C}$ .

A colimit in  $\mathcal{C}$  is  $K$ -absolute if it is preserved by  $\tilde{K}$ . In elementary terms, a colimit  $\sigma_j : P(j) \rightarrow \text{colim } P$  of a functor  $P : \mathcal{J} \rightarrow \mathcal{C}$  is  $K$ -absolute if for all objects  $A \in \mathcal{A}$  the transformation  $\mathcal{C}(K(A), P(j)) \rightarrow \mathcal{C}(K(A), \text{colim } P)$  is a colimit in  $\mathbf{Vect}$ .

Now suppose  $K$  is the inclusion of a full subcategory  $\mathcal{A}$  into  $\mathcal{C}$ . Consider a family of functors  $\Phi = \{P_\gamma : \mathcal{J}_\gamma \rightarrow \mathcal{C}\}_{\gamma \in \Gamma}$ . We say that  $\mathcal{C}$  is the closure of  $\mathcal{A}$  under the family  $\Phi$  if there is no proper replete full subcategory  $\mathcal{D}$  of  $\mathcal{C}$  containing (the image of)  $\mathcal{A}$  such that  $\text{colim } P_\gamma \in \mathcal{D}$  whenever  $P_\gamma$  takes factors through  $\mathcal{D}$ . If  $\Phi$  is the family of all the functors with small (finite) domain into  $\mathcal{C}$ , we say that  $\mathcal{C}$  is the closure of  $\mathcal{A}$  under small (finite) colimits.

We say that  $\Phi$  is a *density presentation* for  $K$  if the colimit of each  $P_\gamma$  exists and is  $K$ -absolute, and  $\mathcal{C}$  is the closure of  $\mathcal{A}$  under the family  $\Phi$ . The functor  $K$  is *dense* when it has a density presentation (although density can be defined in other ways, our choice here is justified by [16, Theorem 5.35]).

Write  $\text{Cocts}^K[\mathcal{C}, \mathcal{B}]$  for the full sub- $\mathbb{k}$ -category of  $[\mathcal{C}, \mathcal{B}]$  of those functors that preserve  $K$ -absolute colimits. The following is a particular instance of [16, Thm. 5.31].

**Theorem 10.** *Let  $\Phi = \{P_\gamma : \mathcal{J}_\gamma \rightarrow \mathcal{C}\}_{\gamma \in \Gamma}$  be a density presentation of the fully faithful functor  $K : \mathcal{A} \rightarrow \mathcal{C}$ . Suppose each  $\mathcal{J}_\gamma$  is small and that  $\mathcal{B}$  admits all small colimits. Then precomposing with  $K$  yields an equivalence*

$$\text{Cocts}^K[\mathcal{C}, \mathcal{B}] \simeq [\mathcal{A}, \mathcal{B}]$$

*with pseudoinverse given by taking left Kan extensions along  $K$ .*

A basic example of dense subcategory is provided by the category of modules over a ring  $R$ . If we take  $\mathcal{C} = {}_R\mathcal{M}$  and  $\mathcal{A}$  the full subcategory determined by the  $R$ -module  $R$ , then  $\mathcal{A}$  is dense in  $\mathcal{C}$ . A density presentation is given by the family of functors with small domain into  ${}_R\mathcal{M}$ . This is so because all colimits are  $K$ -absolute: after identifying  $[\mathcal{A}^{\text{op}}, \mathbf{Vect}]$  with  ${}_R\mathcal{M}$ ,  $K$  is isomorphic to the identity functor. The category  $\mathcal{A}$  is also dense in the category  ${}_R\mathcal{M}_{\text{fp}}$  of finitely presented  $R$ -modules. Indeed,  ${}_R\mathcal{M}_{\text{fp}}$  is the closure of  $\mathcal{A}$  under finite colimits and these are  $K$ -absolute, where  $K$  is the inclusion of  $\mathcal{A}$  into  ${}_R\mathcal{M}_{\text{fp}}$ . Observe that  $\tilde{K}$  in this case is isomorphic to the inclusion of  ${}_R\mathcal{M}_{\text{fp}}$  into  ${}_R\mathcal{M}$ , and hence it preserves colimits. As a consequence of Theorem 10 we have equivalences

$$\text{Cocts}[{}_R\mathcal{M}, \mathcal{B}] \simeq [\mathcal{A}, \mathcal{B}] = R^{\text{op}}\text{-}\mathcal{B} \quad \text{Rex}[{}_R\mathcal{M}_{\text{fp}}, \mathcal{D}] \simeq [\mathcal{A}, \mathcal{D}] = R^{\text{op}}\text{-}\mathcal{D}$$

for any categories  $\mathcal{B}$  and  $\mathcal{D}$  with small colimits and finite colimits respectively. Here  $R^{\text{op}}\text{-}\mathcal{B}$  denotes the category with objects  $B$  of  $\mathcal{B}$  equipped with an action or  $R^{\text{op}}$ , that is, a ring morphism  $R^{\text{op}} \rightarrow \mathcal{B}(B, B)$ , and evident morphisms.

Slightly more general, the Yoneda embedding  $K : \mathcal{A} \rightarrow [\mathcal{A}^{\text{op}}, \mathbf{Vect}]$  is dense for any small category  $\mathcal{A}$ .

A second example of interest for us is subcategory of finite-dimensional comodules  $\mathcal{M}_f^C$ . Let  $K : \mathcal{M}_f^C \rightarrow \mathcal{M}^C$  be the inclusion functor. Given a  $C$ -comodule  $M$ , consider the *comma category*  $K/M$ . That is, the category whose objects are pairs  $(N, f)$  where  $N \in \mathcal{M}_f^C$  and  $f : N \rightarrow M$ , and whose arrows  $(N, f) \rightarrow (N', f')$  are the arrows  $g : N \rightarrow N'$  such that  $f'g = f$ . The functor  $P_M : K/M \rightarrow \mathcal{M}^C$  sending  $(N, f)$  to its  $N$  is the base of a cone of vertex  $M$ , with components  $\sigma_{(N, f)} = f : N \rightarrow M$ . Clearly  $K/M$  is small and filtered (since  $\mathcal{M}_f^C$  has finite colimits and  $K$  preserves them) and  $\sigma$  is a colimiting cone. The family of functors  $P_M$  with  $M$  a  $C$ -comodule is a density presentation for  $K$ : clearly  $\mathcal{M}^C$  is the closure of  $\mathcal{M}_f^C$  under filtered colimits and filtered colimits are preserved by  $\tilde{K}$  since finite dimensional comodules are finitely presentable (*i.e.*,  $\mathcal{M}^C(N, -)$  preserves filtered colimits whenever  $N$  is finite-dimensional). Since  $K$  preserves finite colimits, it is clear that the image of  $\tilde{K} : \mathcal{M}^C \rightarrow [(\mathcal{M}_f^C)^{\text{op}}, \mathbf{Vect}]$  lies in the full subcategory  $\text{Lex}[(\mathcal{M}_f^C)^{\text{op}}, \mathbf{Vect}]$  of left exact functors; moreover, the replete image of  $\tilde{K}$  can be shown to be exactly this subcategory. This yields an equivalence

$$\text{Fin}[\mathcal{M}^C, \mathcal{B}] \simeq [\mathcal{M}_f^C, \mathcal{B}]$$

for any category  $\mathcal{B}$  with filtered colimits, where the category on the left hand side is the category of finitary (*i.e.*, filtered colimit-preserving) functors.

Our next example is the full subcategory  $\mathcal{A}$  of  $\mathcal{M}^C$  determined by the regular comodule  $C$ . Recall that each comodule  $M$  is the equalizer of the canonical pair of comodule morphisms between free comodules  $M \otimes C \rightarrow M \otimes C \otimes C$ . Clearly,  $\mathcal{M}^C$  is the closure of  $\mathcal{A}$  under small limits. For, every free comodule has to be in the closure under small limits, and hence each comodule does too. Now consider the inclusion functor  $K : \mathcal{A}^{\text{op}} \hookrightarrow (\mathcal{M}^C)^{\text{op}}$ . We shall show that the functor  $\tilde{K} : (\mathcal{M}^C)^{\text{op}} \rightarrow [\mathcal{A}, \mathbf{Vect}]$  preserves small colimits. To do this, it is enough to show that the composition of  $\tilde{K}$  with the “forgetful” functor  $[\mathcal{A}, \mathbf{Vect}] \rightarrow \mathbf{Vect}$  (recall that  $\mathcal{A}$  has just one object) preserves small colimits. The resulting functor  $(\mathcal{M}^C)^{\text{op}} \rightarrow \mathbf{Vect}$  is simply  $\mathcal{M}^C(-, C)$ , which is isomorphic to  $\mathbf{Vect}(U(-), \mathbb{k})$ , where  $U$  denotes the forgetful functor  $\mathcal{M}^C \rightarrow \mathbf{Vect}$ . Clearly  $U$  preserves limits and  $\mathbf{Vect}(-, \mathbb{k})$  converts them into colimits. We have shown, then, that small colimits in  $(\mathcal{M}^C)^{\text{op}}$  are  $K$ -absolute. This, together with the fact that  $(\mathcal{M}^C)^{\text{op}}$  is the completion of  $\mathcal{A}^{\text{op}}$  under small colimits, shows that  $K$  is dense. We get, then, an equivalence

$$\text{Cts}[\mathcal{M}^C, \mathcal{B}] \simeq [\mathcal{A}, \mathcal{B}]$$

for any category  $\mathcal{B}$  with small limits; here the category on the left hand side is the category of continuous (*i.e.* small limit-preserving) functors and transformations between them.

Finally, for a finite dimensional coalgebra  $C$ , consider the full subcategory  $\mathcal{A}$  of  $\mathcal{M}^C$  given by the single object  $C$ . As  $(\mathcal{M}_f^C)^{\text{op}}$  is equivalent to  $(\mathcal{M}_{C^\vee})_f$ , the functors  $\mathcal{J} \rightarrow \mathcal{A}^{\text{op}}$  form a finite category  $\mathcal{J}$  constitute a density presentation for  $\mathcal{A}^{\text{op}} \hookrightarrow (\mathcal{M}_f^C)^{\text{op}}$ . Furthermore, we have equivalences

$$\text{Lex}[\mathcal{M}_f^C, \mathcal{B}] \cong \text{Rex}[(\mathcal{M}_f^C)^{\text{op}}, \mathcal{B}^{\text{op}}] \simeq [\mathcal{A}^{\text{op}}, \mathcal{B}^{\text{op}}] \cong [\mathcal{A}, \mathcal{B}]$$

for any category  $\mathcal{B}$  with finite limits. Recall that there is an equivalence  $[\mathcal{A}, \mathbf{Vect}] \cong \mathcal{M}_{C^\vee}$ . For, to give a functor  $\mathcal{A} \rightarrow \mathbf{Vect}$  is to give a vector space with a left action of the algebra  $\mathcal{M}^C(C, C)$ , which is isomorphic to  $(C^\vee)^{\text{op}}$  via the map sending

$\gamma : C \rightarrow \mathbb{k}$  to  $(\gamma \otimes \text{id})\Delta$ . Now it is easy to deduce that for a finite-dimensional coalgebra  $C$  there are equivalences  $\text{Lex}[\mathcal{M}_f^C, \mathcal{M}^D] \simeq {}^C\mathcal{M}^D$ , sending a right exact functor  $F$  to the bicomodule  $F(C)$ . A pseudoinverse for this equivalence is the functor sending a bicomodule  $M$  to  $-\square_C M$ .

In Section 2 we used the following easy observation.

**Observation 22.** Let  $C$  be a finite dimensional coalgebra,  $K : \mathcal{M}_f^C \rightarrow \mathcal{M}^C$  be the inclusion functor and  $M, N \in {}^C\mathcal{M}^D$ . Then we have a string of canonical isomorphisms

$$\begin{aligned} {}^C\mathcal{M}^D(M, N) &\cong \text{Lex}[\mathcal{M}_f^C, \mathcal{M}^D](K(-)\square_C M, K(-)\square_C N) \\ &= [\mathcal{M}_f^C, \mathcal{M}^D](K(-)\square_C M, K(-)\square_C N) \\ &\cong \text{Fin}[\mathcal{M}^C, \mathcal{M}^D](-\square_C M, -\square_C N) \\ &= [\mathcal{M}^C, \mathcal{M}^D](-\square_C M, -\square_C N) \end{aligned}$$

The last piece of categorical background we will need is the *tensor product* of categories with finite limits. This is closely related to Deligne's tensor product of abelian categories of [5]. We only need the case of categories of finite dimensional comodules over finite dimensional coalgebras, though, and in this case the existence of this product reduces to few simple observations.

Recall that the category  $\mathcal{C} \otimes \mathcal{D}$  has as objects pairs of objects  $(c, d)$  with  $c \in \mathcal{C}$  and  $d \in \mathcal{D}$ , and  $\text{homs } \mathcal{C} \otimes \mathcal{D}((c, d), (c', d')) = \mathcal{C}(c, c') \otimes_{\mathbb{k}} \mathcal{D}(d, d')$ . If  $\mathcal{C}, \mathcal{D}, \mathcal{E}$  are categories with finite limits, a functor  $F : \mathcal{C} \otimes \mathcal{D} \rightarrow \mathcal{E}$  is *left exact in each variable* if for each  $c \in \mathcal{C}$  and  $d \in \mathcal{D}$  the functors  $F(c, -) : \mathcal{D} \rightarrow \mathcal{E}$  and  $F(-, d) : \mathcal{C} \rightarrow \mathcal{E}$  are left exact. These functors, together with the natural transformations between them, form a category  $\text{Lex}[\mathcal{C}, \mathcal{D}; \mathcal{E}]$ .

A tensor product of  $\mathcal{C}$  with  $\mathcal{D}$  as categories with finite limits is a category with finite limits  $\mathcal{C} \boxtimes \mathcal{D}$  together with a functor  $\mathcal{C} \otimes \mathcal{D} \rightarrow \mathcal{C} \boxtimes \mathcal{D}$  left exact in each variable that induces equivalences  $\text{Lex}[\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}] \simeq \text{Lex}[\mathcal{C}, \mathcal{D}; \mathcal{E}]$  for each  $\mathcal{E}$ .

The case of interest for us in this work is the one of categories of finite dimensional comodules over finite dimensional coalgebras, dual to the case considered in [5]. If  $C, D$  are finite dimensional coalgebras, we claim that the functor  $\otimes_{\mathbb{k}} : \mathcal{M}_f^C \otimes \mathcal{M}_f^D \rightarrow \mathcal{M}_f^{C \otimes D}$  is a tensor product of  $\mathcal{M}_f^C$  with  $\mathcal{M}_f^D$  as categories with finite limits. To see this, let us call  $\mathcal{C} \subset \mathcal{M}_f^C, \mathcal{D} \subset \mathcal{M}_f^D$  and  $\mathcal{B} \subset \mathcal{M}_f^{C \otimes D}$  the full subcategories determined by the respective regular comodule, and observe that there is a commutative diagram as depicted below.

$$\begin{array}{ccc} \text{Lex}[\mathcal{M}_f^{C \otimes D}, \mathcal{E}] & \longrightarrow & \text{Lex}[\mathcal{M}_f^C, \mathcal{M}_f^D; \mathcal{E}] \\ \simeq \downarrow & & \downarrow \simeq \\ [\mathcal{B}, \mathcal{E}] & \longrightarrow & [\mathcal{C} \otimes \mathcal{D}, \mathcal{E}] \end{array}$$

The horizontal arrows are induced by  $\otimes_{\mathbb{k}}$  and the obvious functor  $\mathcal{C} \otimes \mathcal{D} \rightarrow \mathcal{B}$  which at the level of the unique hom-space is just  $\mathcal{M}_f^C(C, C) \otimes \mathcal{M}_f^D(D, D) \rightarrow \mathcal{M}_f^{C \otimes D}(C \otimes D, C \otimes D)$ . This last linear transformation is an isomorphism, as a consequence of the finiteness of  $C$  and  $D$ , and then the bottom row of the diagram is an isomorphism. It follows that the top row is an equivalence.



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